DEMYSTIFYING COMPACTNESS

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Abstract. We explain how to think about and use the open cover definition of compactness through several examples. In particular, we argue that compactness should be thought of as a finiteness condition which is algorithmic in nature.

Compactness is a fundamental topological property whose definition appears bizarre at first sight.

Definition 1. A topological space is *compact* if every open cover of X admits a finite subcover. In other words, if $\{U_i\}_{i \in I}$ is a collection of open sets of X such that $X = \bigcup_{i \in I} U_i$, then there exists a finite subset $J \subseteq I$ such that $X = \bigcup_{j \in J} U_j$.

This is not a "visual" definition in any sense, but that is the point: fundamentally, compactness is a finiteness condition which is *algorithmic* in nature. Now, compactness admits more visually intuitive characterizations in the special case that $X = \mathbf{R}^n$ and more generally when X is a metric space.

Theorem 2 (Heine-Borel). *A subset $A \subseteq \mathbf{R}^n$ is compact if and only if it is closed and bounded.*

Theorem 3 (Sequential Compactness and Compactness). *For a metric space (X, d), the following are equivalent:*

(a) X is compact with respect to the topology induced by the metric.
(b) X is sequentially compact, i.e., every sequence has a convergent subsequence.

These theorems are very useful and give us intuitive descriptions of compactness in our usual mental models of topological spaces. However, they should really be thought of as theorems about $\mathbf{R}^n$ and about metric spaces rather than as essential characterizations of compactness. After all, there are plenty of topological spaces which are not metric spaces in which the notion of compactness is still meaningful.

What does "algorithmic" mean in this context? Practically speaking, compactness is often used in the following way. Let X be a topological space, and suppose P is a property pertaining to the open sets of X. Suppose X satisfies the following condition:

For each $x \in X$, there exists an open set $U_x \ni x$ of X such that U_x has property P.

Then $X = \bigcup_{x \in X} U_x$, since each $x \in X$ lies in U_x by construction. Compactness then guarantees the existence of a finite collection of open sets $U_1, \ldots, U_n$ such that each U_i has property P and $X = U_1 \cup \cdots \cup U_n$. In this sense, compactness can be viewed as a guarantee that any attempt to patch together a "local property" into a global one will only take a *finite* number of steps (even if this attempt proves unsuccessful). Let's illustrate what I mean with a few easy examples.

Lemma 4. *Let X be a compact topological space. Then every continuous function $f : X \to \mathbf{R}$ is bounded.*

Proof. Since f is continuous, at each point $x \in X$, there exists an open set $U_x \ni x$ such that f is bounded on U_x by some real number $M_x \geq 0$. For example, if $f(x) = y$, take $U_x = f^{-1}((y-1, y+1))$;

on this choice of U_x, f is bounded by $M_x = \max\{|y-1|, |y+1|\}$. By compactness, there exists a finite collection of open sets $U_1, \dots, U_n$ of open subsets of X such that $X = U_1 \cup \cdots \cup U_n$. By construction f is bounded on each U_i by some real number $M_i \geq 0$. Take $M = \max\{M_1, \dots, M_n\}$. Then $|f(x)| \leq M$ for all $x \in X$. □

The point is that every continuous function is *locally* bounded; this much is obvious. In general, for every point $x \in X$, we can always find a small neighborhood of x around which f is bounded by some M_x. What we would like to do is to take as a global bound the number $\max_{x\in X} M_x$, but there is no guarantee that this number is finite, since there could be infinitely many points $x \in X$. Compactness guarantees that we can take the maximum over only finitely many points.

A slightly harder example is the following:

Lemma 5. *Let (X, d) be a compact metric space and $f_n : X \to \mathbf{R}$ be a sequence of continuous functions which is monotonically decreasing, i.e., for each $x \in X$, $m \geq n$ implies $f_m(x) \leq f_n(x)$. If f_n converges pointwise to a continuous function f, then this convergence is uniform.*

Proof. Let $\varepsilon > 0$. We first claim that at each $x \in X$, there is an integer n_x and an open set $U_x \ni x$ such that $f_{n_x}(y) - f(y) < \varepsilon$ for all $y \in U_x$. By pointwise convergence, there exists an integer n_x such that $f_{n_x}(x) - f(x) < \varepsilon/3$. By continuity of f_{n_x} and f, there exists a $\delta > 0$ such that for all $y \in X$ with $d(y, x) < \delta$, we have both $|f_{n_x}(y) - f_{n_x}(x)| < \varepsilon/3$ and $|f(y) - f(x)| < \varepsilon/3$. Take $U_x = B(x, \delta)$, the open ball of radius δ around x. Then for all $y \in U_x$, we have

$$f_{n_x}(y) - f(y) \leq |f_{n_x}(y) - f_{n_x}(x)| + |f_{n_x}(x) - f(x)| + |f(x) - f(y)| < 3 \cdot \frac{\varepsilon}{3} = \varepsilon,$$

thus establishing the existence of U_x. Note that since the f_i are monotonically decreasing, this also implies that $f_m(y) - f(y) < \varepsilon$ for all $y \in U_x$ and $m \geq n_x$.

Now by compactness, there are a finite number of open sets $U_1, \dots, U_r$ and integers $n_1, \dots, n_r$ such that $X = U_1 \cup \cdots \cup U_r$ and for all $y \in U_j$, we have $f_{n_j}(y) - f(y) < \varepsilon$. Taking $n = \max_j n_j$, we have $f_m(x) - f(x) < \varepsilon$ for all $m \geq n$ and $x \in X$, which implies $f_m \to f$ uniformly. □

Note that the compactness hypothesis is crucial. Consider, for example, the functions $f_n : \mathbf{R} \to \mathbf{R}$ defined by

$$f_n(x) = \begin{cases} e^{-n/x^2} & x \neq 0, \\ 0 & x = 0. \end{cases}$$

Then f_n is a monotonically decreasing sequence of functions converging pointwise, but not uniformly, to 0.

In both of the above examples, we began by establishing a condition which was straightforward to verify locally; compactness was then the hypothesis we needed to patch it together globally. Note that the *finiteness* was the key. As another demonstration of this idea, let's give a proof that compactness implies sequential compactness:

Theorem 6. *Let (X, d) be a compact metric space. Then every sequence has a convergent subsequence.*

Proof. Let $\{x_n\}$ be an arbitrary sequence in X and suppose by contradiction that it has no convergent subsequence. Then for any point $y \in X$, the fact that no subsequence of $\{x_n\}$ converges to y implies the existence of a small enough open set U_y and n_y such that $\{x_n\}$ "eventually stops visiting U_y," i.e. $x_n \notin U_y$ for all $n \geq n_x$. Since X is compact, a finite number of these opens $U_{y_1}, \dots, U_{y_m}$ cover X. Take $m = \max\{n_{y_i}\}$. By assumption, $x_n \notin U_{y_i}$ for any $n \geq m$. But then $x_m \notin \bigcup U_{y_i} = X$, a contradiction. □

The point is that if $\{x_n\}$ has no convergent subsequence, then it must "stop visiting" a neighborhood of every point. If the space is compact, then there are only finitely many relevant neighborhoods, so it must eventually leaves every neighborhood in finite time, i.e. escape X (which is impossible by assumption). Notice how this proof fails miserably when we take $x_n = 1/n$ in $X = (0, 1)$ with the standard metric and $U_{y_i} = (1/i, 1)$; $\{x_n\}$ eventually escapes every U_{y_i} but can never escape all of them simultaneously within finite time.

Compactness is a condition which requires some practice to be accustomed to; it is a notion which pertains to a proof on a "meta-mathematical" level, an assurance of finiteness in an algorithmic sense. It is no surprise then that compactness ends up being a useful finiteness condition to impose in a variety of circumstances, even when the topology of the underlying space is strange or exotic.