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Exploring Parking Functions: Poset and Polytope Perspectives

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Abstract

This paper provides an exploration of parking functions, a classical combinatorial object. We present two viewpoints on their structure and properties: one through the poset of noncrossing partitions and the other through polytopes.

1. Parking Functions

1.1. *Parking functions.* Imagine living on a one-way street that dead-ends with n parking spots available. You and your neighbors have n cars in total, and everyone has their preferred spot to park. Without reversing, does there exist a solution that everyone can park without collision? In mathematics, this real life dilemma is called the parking problem. To formalize this problem, let's consider below four conditions.

a. Let $n \in \mathbb{Z}^+$. On a one-way street, there are n parking spots labeled as $1, 2, \ldots, n$ and n cars want to park here.
b. Car C_i is the i-th car to park, which has preferred parking spot $\alpha_i \in [n] = \{1, 2, \ldots, n\}$. More than one car can have the same preference.
c. If the preferred spot of some car had already been occupied, the car will move forward and park in the next available spot.
d. No backward movement is allowed.

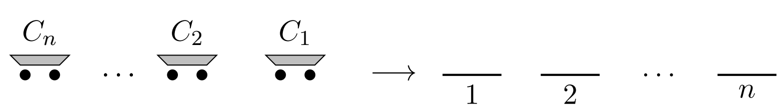

Figure 1. Parking process illustration.

It is not guaranteed that every car will be able to park before driving to the dead-end. If all n cars are able to park under the above conditions, then we

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say the preference list $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n)$ is a *parking function*. We use PF_n to denote the set of all parking functions of length n. For simplicity, sometimes we drop the parentheses and commas in the preference list. For example, all parking functions of length 3 are:

$$\begin{array}{cccccccc} 111, & 112, & 121, & 211, & 113, & 131, & 311, & 122, \\ 212, & 221, & 123, & 132, & 213, & 231, & 312, & 321. \end{array}$$

On the contrary, 322 is not a parking function of length 3.

It is very natural to ask: how many parking functions are there? In 1966, Konheim and Weiss counted the number of parking functions.

Theorem 1.1 ([KW66]). *For a positive integer n, $|\mathrm{PF}_n| = (n+1)^{n-1}$.*

Proof. In this proof, we adopt Pollak's approach (see [Rio69]): We add an additional spot $n+1$ and arrange the spots in a circle clockwise. We also allow $n+1$ to be chosen as a preferred spot. With this circular arrangement, all cars are able to park because they can circle around until an available spot is found. For example, consider the preference list $(4, 4, 1, 3)$. On the one way street, C_2 cannot park. However, as illustrate in Figure 2a, C_2 can park at spot 5 in this circular arrangement.

Say the preference list in this set up is a *circular parking function* of length n, and we use CPF_n to denote the set of all circular parking functions of length n. Since each of the n cars has $n+1$ choices for its preferred spot, we have

$$|\mathrm{CPF}_n| = (n+1)^n.$$

Observe that any circular parking function leaves one spot empty, say the label of the empty spot is $k \in [n+1]$. We relabel the $n+1$ spots in the circle by assigning the label 0 to the empty spot, the label 1 to the next spot in the clockwise direction, and so on. Given a circular parking function $\gamma = (\gamma_1, \dots, \gamma_n)$, we can "unwrap" it to a parking function $\alpha = (\alpha_1, \dots, \alpha_n)$ by shifting each preference down by k, i.e., we set $\alpha_i = \gamma_i - k \pmod{n+1}$ for each $i \in [n]$.

We say two circular parking functions $\gamma_1, \gamma_2 \in \mathrm{CPF}_n$ are *equivalent* if they are related by a circular rotation, i.e.,

$$\gamma_1 \equiv \gamma_2 - k \pmod{n+1}$$

for $k \in [n+1]$. This relation defines an equivalence class, and each equivalence class contains $n+1$ circular parking functions, one for each possible rotation. Since all circular parking functions in the same equivalence class unwrap to the same parking function, we conclude

$$|\mathrm{PF}_n| = \frac{|\mathrm{CPF}_n|}{n+1} = \frac{(n+1)^n}{n+1} = (n+1)^{n-1}. \qquad \square$$

(a) $\gamma = (4, 4, 1, 3)$ (b) $\alpha = (2, 2, 4, 1)$

$$\alpha_i = \gamma_i - 2 \pmod 5, \text{ where } i = 1, 2, 3, 4.$$

Figure 2. Circular parking function γ to parking function α.

Parking functions have many nice properties. For example, among the parking functions of length 3, observe that some are rearrangements of the other. In general, we have the below lemma.

Lemma 1.2. *Every permutation of the entries of a parking function is also a parking function.*

Proof. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a parking function of length n. Any rearrangement of α can be obtained by successively swapping two adjacent entries one at a time (see Figure 3 for an example). Thus, it suffices to show that if $\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_{i+1}, \alpha_i, \ldots, \alpha_n)$ is obtained by swapping the ith and $(i+1)$th entries of α, for each $i \in [n-1]$, then α' is also a parking function.

$$(2, 2, 1, 4, 3) \to (2, 1, 2, 4, 3) \to (1, 2, 2, 4, 3) \to (1, 2, 2, 3, 4)$$

Figure 3. Rearranging $(2, 2, 1, 4, 3)$ into nondecreasing order.

Let C_i and C_{i+1} denote the cars whose preferences are α_i and α_{i+1} in α, respectively. The new preference list α' modifies the order in which cars park. C_i becomes the $(i+1)$-th car and desires α'_{i+1} in α' (which is α_i in α). To avoid ambiguity, we denote C_i as C'_{i+1} under α'. Similarly, C_{i+1} becomes the i-th car in α', desiring α'_i (which is α_{i+1} in α), and we denote it as C'_i.

Assume that under the original preference list α, cars C_i and C_{i+1} park at spots x and y, respectively, with $x, y \in [n]$. Consequently, we have $\alpha_i \leq x$ and $\alpha_{i+1} \leq y$. We consider three cases: $\alpha_i = \alpha_{i+1}$, $\alpha_i < \alpha_{i+1}$, and $\alpha_i > \alpha_{i+1}$. In the first case, we have $\alpha = \alpha'$, so α' is a parking function trivially. In the rest of this proof, we focus on proving the second case, $\alpha_i < \alpha_{i+1}$. The third case is similar to the second one and is left to the reader as an exercise.

If $\alpha_i < \alpha_{i+1}$, it follows that $x < y$. When C'_i is about to park, spot x and y are still available. If $x < \alpha_{i+1}$, C'_i initially attempts to park at its preferred spot $\alpha'_i = \alpha_{i+1}$, which exceeds x. The next available spot after α'_i is y (which may coincide with α'_i), so C'_i parks at y. Next, when C'_{i+1} is ready to park, it prefers the spot $\alpha'_{i+1} = \alpha_i$. Since x is still available, C'_{i+1} parks at x, which is either its preferred spot or the next available spot after α'_{i+1}. See Figure 4. Similarly, if $\alpha_{i+1} \leq x$, then C'_i parks at x and C'_{i+1} parks at y.

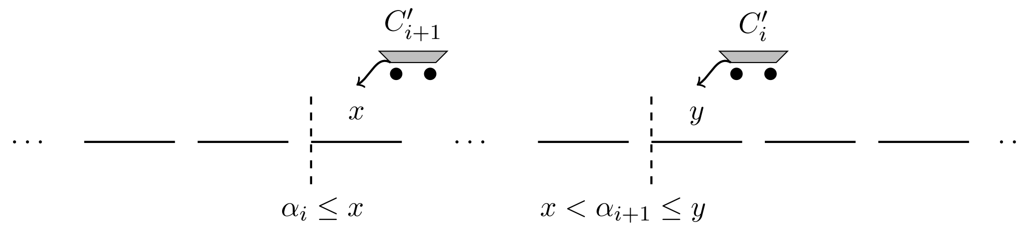


Figure 4. Under the preference list α', C'_{i+1} parks at x and C'_i parks at y, if $\alpha_i \leq x < \alpha_{i+1}$.

Under the preference list α', the remaining $n-2$ cars park in the same spots as in α. Since both C'_i and C'_{i+1} successfully park in all three cases, we conclude that α' is a parking function. □

The following proposition provides a simple criterion for determining whether a finite sequence is a parking function. Some people ([AW21], [MV23], and [Sta97]) adopt it as the definition of parking functions .

Proposition 1.3. *Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be a finite sequence of positive integers. Let $\beta = (\beta_1, \ldots, \beta_n)$ be a rearrangement of α in nondecreasing order, i.e., $\beta_1 \leq \beta_2 \leq \cdots \leq \beta_n$. Then α is a parking function if and only if $\beta_i \leq i$ for all $i \in [n]$.*

Proof. Suppose $\beta_j > j$ for some $j \in [n]$. Let the leftmost β_j appear at position i in α. This implies that $\alpha_i = \beta_j$, and car C_i desires β_j. After cars $C_1, \ldots, C_{i-1}$ have parked, there remains an empty spot before j since $\beta_j > j$. For cars $C_{i+1}, \ldots, C_n$, they first drive to their preferred spot (which exceeds β_j and j) and move forward if it has already been occupied. Therefore, no car will park at the empty spot before j, i.e., α is not a parking function.

Conversely, suppose $\beta_i \leq i$ for all $i \in [n]$. It is trivial to see that β is a parking function. By Lemma 1.2, we know that α is a parking function. □

1.2. *Primitive parking functions.* A parking function is *primitive* if it is arranged in nondecreasing order. Every parking function can be rearranged into a primitive parking function. For example, the parking function $(3, 2, 1, 3)$ can be rearranged into the primitive parking function $(1, 2, 3, 3)$. The number of primitive parking functions of length n is given by the nth *Catalan number*

$C_n = \frac{1}{n+1}\binom{2n}{n}$. The Catalan numbers are a sequence of positive integers that counts a wide range of combinatorial objects, which are thus called *Catalan objects*. Another Catalan object appears in Theorem 2.3. For a more comprehensive overview of Catalan objects, see [Sta98].

We establish a bijection between the number of primitive parking functions of length n and the nth Catalan number through Dyck paths.

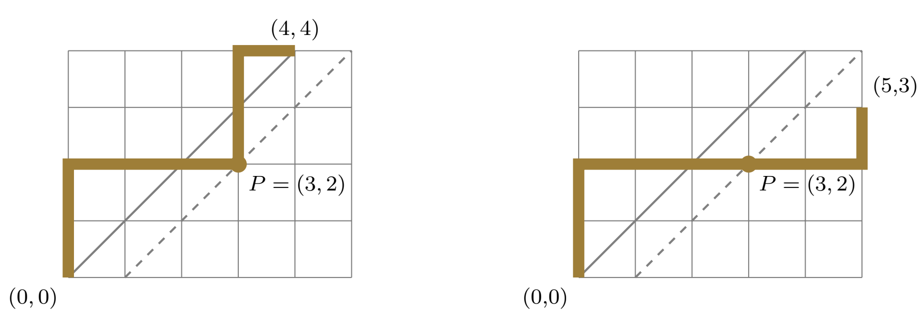
Definition 1.4. A *Dyck path* of length n is a path from $(0,0)$ to (n,n) consisting of *east steps* $(1,0)$ and *north steps* $(0,1)$, such that the path never goes below the diagonal $y = x$, i.e., the path does not contain any point (a,b) with $a > b$.

THEOREM 1.5. *The number of Dyck path of order n is given by the nth Catalan number C_n.*

Proof. Consider the set of paths from $(0,0)$ to (n,n) consisting of east steps and north steps with no restrictions. To get such a path, we must take n steps east and n steps north, so there are $\binom{2n}{n}$ of them. We say a path from $(0,0)$ to (n,n) is *bad* if it crosses below the diagonal $y = x$ at any point, i.e., a bad path contains point (a,b) with $a > b$. Let B_n denote the set of all bad paths from $(0,0)$ to (n,n). Hence, we want to show that

$$C_n = \binom{2n}{n} - |B_n|.$$

Consider any bad path and let $P = (i+1, i)$ be the first point where it crosses below the diagonal $y = x$, where $0 \le i \le n-1$. See Figure 5a. From P, we need $n - i$ east steps and $n - (i+1)$ north steps to reach (n,n). If, instead, we *reflect* the remaining portion by taking $n - i$ north steps and $n - (i+1)$ east steps from P, we will arrive at $(n+1, n-1)$, see Figure 5b.



(a) A bad path in B_4

(b) The reflected path

Figure 5. Path (A) maps to (B) under the bijection, vice versa.

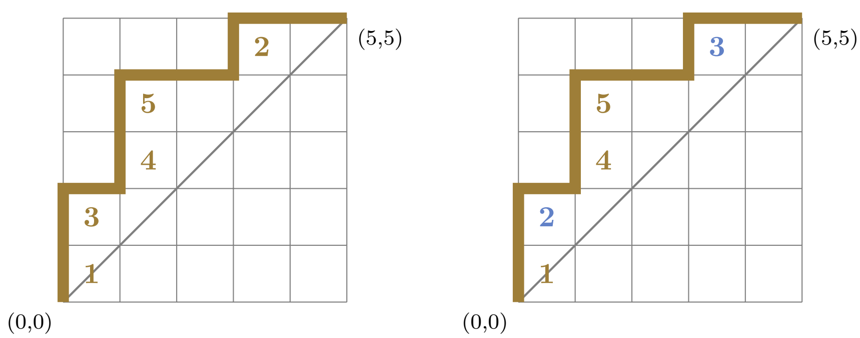
Conversely, for any path from $(0,0)$ to $(n+1, n-1)$, it must contain a point strictly below the diagonal $y = x$. Let $P = (i+1, i)$ be such a point where i is the smallest possible choice and $1 \le i \le n-1$. From P, if we reflect and take $n-i$ steps east, $n-(i+1)$ steps north, the path will end at (n,n). Therefore, we conclude that there is a bijection between all paths from $(0,0)$ to $(n+1, n-1)$ and bad paths from $(0,0)$ to (n,n).

From $(0,0)$ to $(n+1, n-1)$, there are $\binom{n+1+(n-1)}{n+1} = \binom{2n}{n+1}$ distinct paths. Hence, we have

$$C_n = \binom{2n}{n} - \binom{2n}{n+1} = \frac{1}{n+1}\binom{2n}{n}. \qquad \square$$

Dyck paths can be *labeled* following below process. Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a parking function, and let $\beta = (\beta_1, \dots, \beta_n)$ be the primitive parking function obtained by rearranging α in nondecreasing order, i.e., $\beta_1 \le \cdots \le \beta_n$. For each $i \in [n]$, let k_i denote the number of occurrences of i in β. Starting at $(0,0)$, for each i, move k_i steps north and then one step east. Note that this creates a Dyck path of length n. To label this Dyck path, let $j_1, j_2, \dots$ denote the positions in α where i appears. Assign the labels $j_1, j_2, \dots$ to the vertical steps of the Dyck path in column i, ordering them from the bottom to the top. See Figure 6a for an example.

Conversely, given a labeled Dyck path, we can reconstruct the parking function $\alpha = (\alpha_1, \dots, \alpha_n)$ by setting $\alpha_j = i$ if j appears in column i of the labeled Dyck path. Therefore, we have a bijection between parking functions of length n and labeled Dyck paths of length n.



(a) $\alpha = (1,4,1,2,2)$.

(b) $\alpha' = (1,1,4,2,2)$

Figure 6. The labeled Dyck paths of α and α'.

Let α be a parking function and α' a distinct rearrangement of α. Observe that the labeled Dyck paths of α and α' are the same Dyck path but with different labelings, since both α and α' can be sorted into the same primitive parking function. In particular, if α' is obtained from α by swapping the entries

α_i and α_j for some distinct $i, j \in [n]$, then the labeled Dyck path of α' can be obtained by swapping the labels i and j in the labeled of Dyck path of α. See Figure 6b for an example. Consequently, we have a bijection between primitive parking functions of length n and Dyck paths of length n.

In the rest of this paper, we explore the combinatorial structure of parking functions from two perspectives: one from enumerative combinatorics and the other from geometry.

2. Parking Functions and Posets

Combinatorialists are interested in Catalan objects and the bijections among them. In Section 1.2, we showed that primitive parking functions are Catalan objects, which motivates us to explore connections between these and other Catalan objects. In this section, we focus on one of the important Catalan objects: the noncrossing partitions of $[n]$.

Definition 2.1. A *partition of a finite set* S is a collection $\{B_1, B_2, \ldots, B_k\}$ of nonempty pairwise disjoint subsets $B_i \subseteq S$ such that $B_1 \cup B_2 \cup \cdots \cup B_k = S$, where $i \in [k]$. A *noncrossing partition* of $[n]$ is a partition $\{B_1, B_2, \ldots, B_k\}$ of $[n]$ such that if $a, c \in B_i$ and $b, d \in B_j$ with $a < b < c < d$, we have $i = j$.

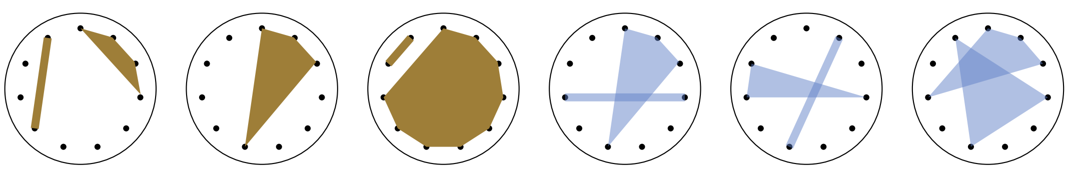


Figure 7. Noncrossing partitions (first three on the left) and crossing partitions (last three on the right) of [11].

Let $\pi = \{B_1, B_2, \ldots, B_k\}$ be a noncrossing partition of $[n]$. Each B_i is called a *block* of π, and we use $|\pi|$ to denote the number of blocks in π. Let NC_n denote the set of all noncrossing partitions of $[n]$. A fundamental property of NC_n is, it can be given a natural partial ordering.

Definition 2.2. A *partially ordered set*, or *poset*, is a pair $(P, \leq)$ where P is a set and "$\leq$" is a binary relation on P satisfying

a. (reflexivity) $x \leq x$,
b. (antisymmetry) if $x \leq y$ and $y \leq x$, then $x = y$, and
c. (transitivity) if $x \leq y$ and $y \leq z$, then $x \leq z$,

for all $x, y, z \in P$.

NC_n can be given a natural partial order such that $\pi \leq \sigma$ if every block of π is contained in a block of σ.

Often a poset is represented by a certain graph which can be easier to work with than just using above axioms. For $x, y \in P$, if $x \leq y$ and $x \neq y$, we write $x < y$. We say x is *covered* by y (or y *covers* x), written either $x \lessdot y$ or $y \gtrdot x$, if $x < y$ and there is no $z \in P$ with $x < z < y$. The *Hasse diagram* of P is the graph with vertices P and an edge from x up to y if $x \lessdot y$. Figure 8 is the Hasse diagram of NC_3, where we have $\pi_1 \lessdot \pi_{2_1} \lessdot \pi_3$, $\pi_1 \lessdot \pi_{2_2} \lessdot \pi_3$, and $\pi_1 \lessdot \pi_{2_3} \lessdot \pi_3$. Observe that two noncrossing partitions are at the same level in the Hasse diagram if they are incomparable with respect to the partial order defined above. Moreover, they partition $[n]$ into the same number of blocks.

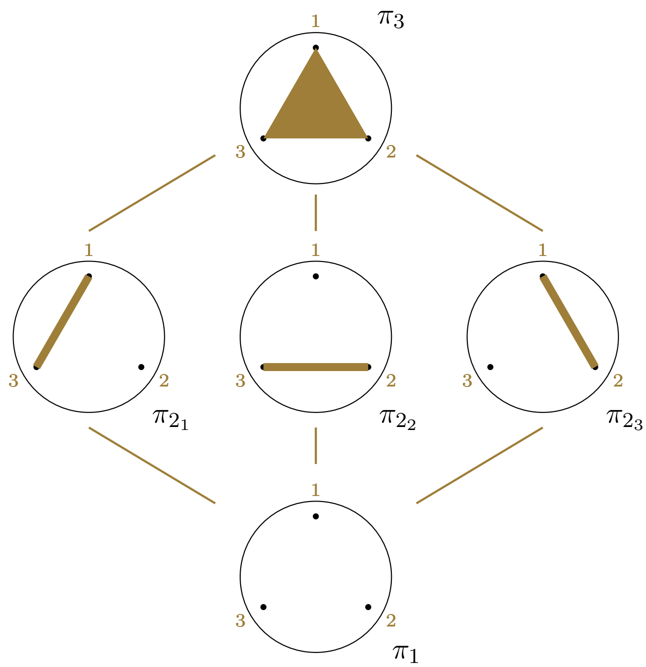


Figure 8. Hasse diagram of NC_3.

Theorem 2.3 ([OB49]). *The number of noncrossing partitions of* $[n]$ *is the nth Catalan number.*

One can prove Theorem 2.3 by showing that $|\,\mathrm{NC}_n\,|$ satisfies the recurrence relation for the Catalan numbers

$$C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i},$$

where we set $C_0 = 1$. For more details, see [OB49, pg. 697]. Therefore, there exists a bijection between these two Catalan objects: the noncrossing partitions of $[n]$, and the primitive parking functions of length n. Primitive parking functions are a special case of parking functions. This motivates us to ask, does there exist a set, related to noncrossing partitions of $[n]$, that corresponds bijectively to the full set of parking function of length n? See Figure 9 for an illustration.

$$\begin{array}{ccc} \{\text{Parking function of length } n\} & \longleftrightarrow & \textbf{what set}? \\ \downarrow & & \downarrow \\ \{\text{Primitive parking functions of length } n\} & \longleftrightarrow & \{\text{Noncrossing partitions of } [n]\} \end{array}$$

Figure 9

In 1997, Richard P. Stanley showed that the number of maximal chains in noncrossing partitions of $[n+1]$ coincides with the number of parking functions of length n providing a bijective correspondence.

Definition 2.4. A *chain* $\mathfrak{c}$ in NC_n is a sequence of noncrossing partitions $\pi_1, \pi_2, \ldots, \pi_r$, where $r \le n$ and $\pi_i \le \pi_{i+1}$ for each $i \in [r-1]$. A *maximal chain* $\mathfrak{m}$ in NC_n is a chain consisting n noncrossing partitions $\pi_1, \pi_2, \ldots, \pi_n$ such that for each $i \in [n-1]$, $\pi_i \le \pi_{i+1}$, and π_{i+1} can be obtained from π_i by merging two blocks of π_i into a single block.

In Figure 8, $\mathfrak{c} = \pi_1, \pi_3$ is a chain in NC_3, and $\mathfrak{m} = \pi_1, \pi_{2_1}, \pi_3$ is a maximal chain in NC_3. A maximal chain in NC_n can be thought of as a series of steps starting with the finest partition $\pi_1 = \{1\}, \{2\}, \ldots, \{n\}$ (i.e., the partition contains only singletons) and then merging two blocks together one at a time at each step until we reach $\pi_n = \{1, 2, \ldots, n\}$. For example, to build the maximal chain $\mathfrak{m} = \pi_1, \pi_{2_1}, \pi_3$ in Figure 8, we merge the blocks $\{1\}$ and $\{3\}$ of π_1 at step 1, merge the blocks $\{1,3\}$ and $\{2\}$ of π_{2_1} at step 2, and we reach $\pi_3 = \{1, 2, 3\}$.

Let $\sigma \in \mathrm{NC}_n$ and let $\mathfrak{m}$ be any maximal chain in NC_n containing σ. The *rank* of σ is the number of merging steps needed to reach σ from the finest partition along $\mathfrak{m}$. We use $\mathrm{rank}(\sigma)$ to denote the rank of σ. For example, in Figure 8, $\mathrm{rank}(\pi_1) = 0$, $\mathrm{rank}(\pi_{2_1}) = \mathrm{rank}(\pi_{2_2}) = \mathrm{rank}(\pi_{2_3}) = 1$, and $\mathrm{rank}(\pi_3) = 2$. Observe that every $\sigma \in \mathrm{NC}_n$ satisfies $0 \le \mathrm{rank}(\sigma) \le n-1$, and $\mathrm{rank}(\sigma) = n - |\sigma|$. Let $r \le n-2$. Given any distinct integers $t_1, t_2, \ldots, t_r$ such that $0 < t_1 < \cdots < t_r < n-1$, let $N(t_1, t_2, \ldots, t_r)$ denote the number of chains $\mathfrak{c} = \sigma_1, \ldots, \sigma_r$ in NC_n such that $\mathrm{rank}(\sigma_i) = t_i$. In 1980, Edelman gave a formula to count $N(t_1, t_2, \ldots, t_r)$.

Proposition 2.5 ([Ede80]). *Set* $t_0 = 0$, $t_{r+1} = m-1$, *and* $s_i = t_i - t_{i-1}$ *for* $1 \leq i \leq r+1$, *Then*

$$N(t_1, t_2, \ldots, t_r) = \frac{1}{n}\binom{n}{s_1}\binom{n}{s_2}\cdots\binom{n}{s_{r+1}}.$$

To prove Proposition 2.5, Edelman exhibited a bijection between an $r+1$-tuple of subsets of $[n]$ with sizes $m - s_1, s_2, \ldots, s_{r+1}$ and a pair $(\mathfrak{c}, b)$ with appropriate properties, where $\mathfrak{c}$ is a chain in NC_n and $b \in [n]$. For more details, see [Ede80, Theorem 3.2]. As an immediate consequence, the number of maximal chains in NC_n is given by

$$N(1, 2, \ldots, n-2) = \frac{1}{n}\underbrace{\binom{n}{1}\binom{n}{1}\cdots\binom{n}{1}}_{n-1 \text{ times}} = n^{n-2}.$$

How about the number of maximal chains in NC_{n+1}? A quick calculation tells us that NC_{n+1} has $(n+1)^{n-1}$ maximal chains, which coincides the number of parking functions of length n [Theorem 1.1]. This motivates Stanley's theorem.

Theorem 2.6 ([Sta97]). *There is a bijection between the maximal chains in* NC_{n+1} *and* PF_n.

We describe Stanley's bijection. Let $\mathfrak{m} = \pi_1, \ldots, \pi_{n+1}$ be a maximal chain in NC_{n+1}. Partition π_{n+1} can be obtained by n merging steps starting with the finest partition π_1 and then merging two blocks together one at a time at each step. Stanley's bijection then gives labels $\alpha_1, \ldots, \alpha_n$ to each step: Let A and B be the two blocks of π_i we are going to merge at step i, where A contains the smallest element in the disjoint union $A \cup B$. The label α_i for this step is the largest element in A which is smaller than all elements in B. The sequence of labels $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ forms a parking function of length n.

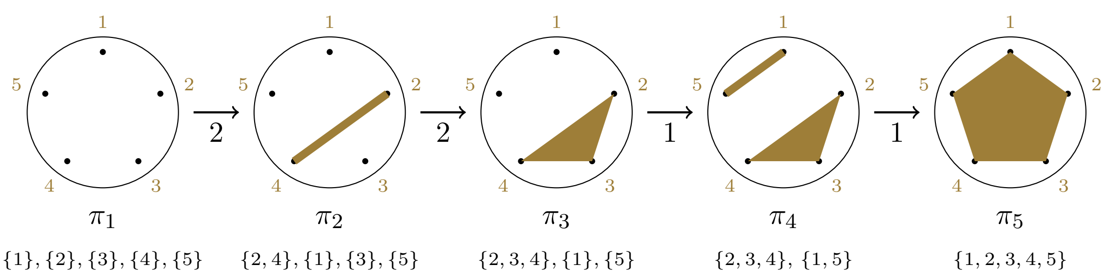


Figure 10. One of the maximal chain in $\{1, 2, 3, 4, 5\}$ which is associated with parking function $(2, 2, 1, 1)$.

For example, consider the maximal chain $\mathfrak{m} = \pi_1, \pi_2, \pi_3, \pi_4, \pi_5$ in NC_5 illustrated in Figure 10. At step 1, we merge the blocks $\{2\}$ and $\{4\}$ of π_1. Let

$A = \{2\}$ and $B = \{4\}$, where $2 \in A$ is the smallest element in $A \cup B = \{2, 4\}$. We then choose $\alpha_1 = 2$ since 2 is the only element in A. At step 2, we merge the blocks $\{2, 4\}$ and $\{3\}$ of π_2. Let $A = \{2, 4\}$ and $B = \{3\}$, where $2 \in A$ is the smallest element in $A \cup B = \{2, 3, 4\}$. We then choose $\alpha_2 = 2$ since 2 is the largest element in A smaller than $3 \in B$. At step 3, we merge the blocks $\{1\}$ and $\{5\}$ of π_3. Let $A = \{1\}$ and $B = \{5\}$, where $1 \in A$ is the smallest element in $A \cup B = \{1, 5\}$. We then choose $\alpha_3 = 1$ since 1 is the only element in A. At step 4, we merge the blocks $\{2, 3, 4\}$ and $\{1, 5\}$ of π_4. Let $A = \{1, 5\}$ and $B = \{2, 3, 4\}$, where $1 \in A$ is the smallest element in $A \cup B = \{1, 2, 3, 4, 5\}$. We then choose $\alpha_4 = 1$ since 1 is the largest element in A smaller than all elements in B.

Proof sketch of Theorem 2.6. Let $\mathfrak{m} = \pi_1, \pi_2, \ldots, \pi_{n+1}$ be a maximal chain in NC_{n+1}. Suppose π_{i+1} is obtained from π_i by merging two blocks A and B of π_i, where $\min A < \min B$ and $i \in [n]$. The Stanley's map labels step i by

$$\alpha_i \coloneqq \max\{j \in A : j < \min B\}.$$

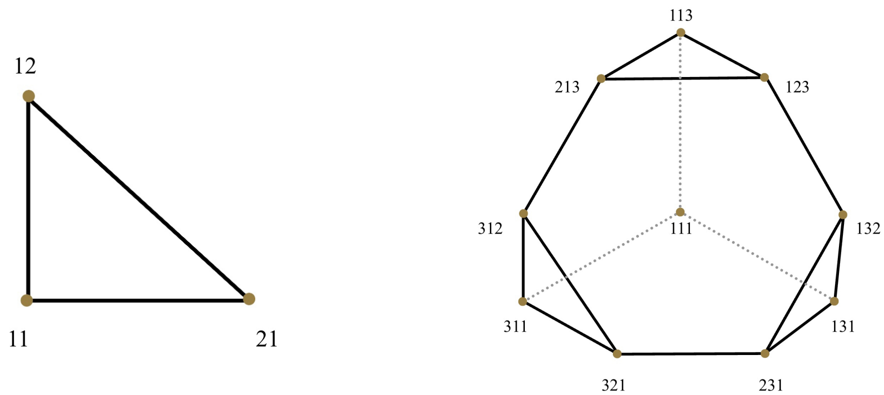
First we want to show that $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a parking function. Note that $\alpha_i < \min B$, which means that if α_i is the label of some step, the merged block of π_{i+1} containing α_i also contains an element $k > \alpha_i$. This ensures that the number of occurrences of α_i in α is at most $n + 1 - \alpha_i$. Indeed, α is a parking function. If α_i appears exactly $n + 1 - \alpha_i$ times, it will occupy the α_i-th to the n-th entries in the nondecreasing rearrangement of α. Therefore, the leftmost α_i in the sorted sequence cannot exceed the position α_i.

We briefly sketch the proof of injectivity. For further details, see [Sta97, Theorem 3.1]. Let $\mathfrak{m} = \pi_1, \ldots, \pi_{n+1}$ be a maximal chain in NC_{n+1} and $\alpha = (\alpha_1, \ldots, \alpha_n)$ be its image under Stanley's map. Let $r = \max\{\alpha_i : 1 \le i \le n\}$ and $s = \max\{i : \alpha_i = r\}$. Suppose π_{s+1} is obtained from π_s by merging two blocks A and B of π_s, where A contains the smallest element in $A \cup B$. We claim that A contains r and $B = \{r + 1\}$. Using this claim and induction on n, we can uniquely recover a chain $\mathfrak{m}$ in NC_{n+1} from a given parking function α of length n.

To show surjectivity, note that NC_{n+1} has $(n + 1)^{n-1}$ maximal chains, which is equal to the number of parking functions of length n. Consequently, each parking function of length n appears exactly once as an image under Stanley's map. □

3. Parking Function Polytopes

Usually, when dealing with a finite integer sequence, a combinatorialist first examines its discrete properties rather than interpreting it geometrically. However, when parking functions of length n are placed in $\mathbb{R}^n$, the resulting structure is surprisingly organized: the points are not arbitrarily scattered but instead form a polytope that exhibits a remarkably rich combinatorial structure. For example, the polytope formed by parking functions of length 2–$(1,1)$, $(1,2)$, and $(2,1)$–is a triangle; see Figure 11a. This section aims to explore the geometric structure of polytopes formed by parking functions.



(a) Parking function polytope $\mathcal{PF}_2$.

(b) Parking function polytope $\mathcal{PF}_3$.

Figure 11

First, we recall some definitions. A point set $K \subseteq \mathbb{R}^n$ is *convex* if for any two points $\mathbf{x}, \mathbf{y} \in K$, it contains a straight line segment between them. For example, the set in Figure 12a is convex, while the set in Figure 12b is not convex since it does not contain the line segment connecting A and B. The *convex hull* of K, denoted by $\operatorname{conv}(K)$, is the smallest convex set containing K, and can be constructed as the intersection of all convex sets that contains K. Namely,

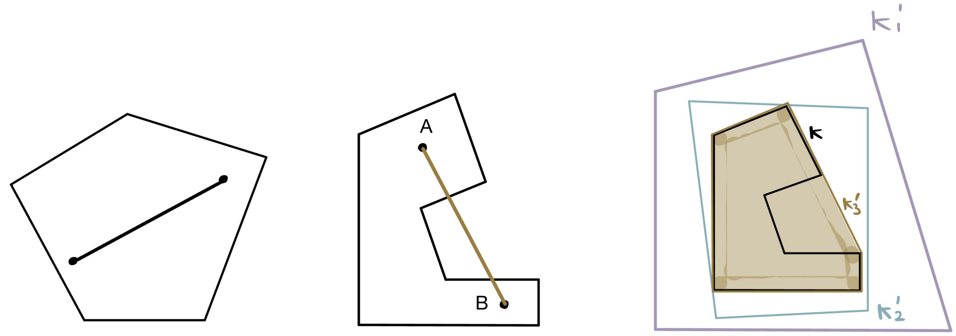
$$\operatorname{conv}(K) = \bigcap \{K' \subseteq \mathbb{R}^n : K \subseteq K', K' \text{ is convex}\}.$$

For example, in Figure 12c, the sets K_1', K_2' and K_3' are convex sets containing K, and the shaded area is $\operatorname{conv}(K)$.

Let $K = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\} \subseteq \mathbb{R}^n$ be a finite point set. A *convex combination* of K is any vector $\mathbf{x} \in \mathbb{R}^n$ of the form

$$\mathbf{x} = \lambda_1 \mathbf{v}_1 + \lambda_2 \mathbf{v}_2 + \cdots + \lambda_k \mathbf{v}_k,$$

where $\lambda_i \geq 0$ for all i, and $\lambda_1 + \cdots + \lambda_k = 1$. A *polytope* is the convex hull of a finite point set. In other words, for a given finite point set $K \subseteq \mathbb{R}^n$, the corresponding polytope consists precisely of those points that can be expressed as convex combinations of the points in K.



(a) A convex set. (b) Not a convex set. (c) Convex hull of K.

Figure 12

Definition 3.1. The *parking function polytope* $\mathcal{PF}_n \subseteq \mathbb{R}^n$ is the convex hull of all parking functions of length n, i.e.,

$$\mathcal{PF}_n = \operatorname{conv}\left(\{\alpha \in \mathbb{R}^n : \alpha \text{ is a parking function of length } n\}\right).$$

Understanding the number of faces of a polytope is crucial for analyzing its geometric structure. Let $\mathcal{P} \subseteq \mathbb{R}^n$ be a polytope. A subset $F \subseteq \mathbb{R}^n$ is a *face* of $\mathcal{P}$ if $F = \mathcal{P} \cap \{\mathbf{x} : \mathbf{c} \cdot \mathbf{x} = d\}$ for some $\mathbf{c} \in \mathbb{R}^n$ and $d \in \mathbb{R}$, such that for all $\mathbf{x} \in \mathcal{P}$, $\mathbf{c} \cdot \mathbf{x} \leq d$, where the dot $\cdot$ means the dot product. The *dimension* of F is the dimension of the smallest affine subspace of $\mathbb{R}^n$ containing F. We call a face a *vertex* if it has dimension 0, an *edge* if it has dimension 1, and a *facet* if it has dimension $n-1$, where $\mathcal{P}$ has dimension n. Each parking function of length n lies on some face of $\mathcal{PF}_n$, whether the face is a vertex, an edge, or a higher-dimensional face. $\mathcal{PF}_3$ has 7 facets: one hexagonal facet, three pentagonal facets, and three triangular facets, see Figure 11b. The hexagonal facet, known as permutahedron Π_3, is particularly notable.

Π_3 was first investigated by Schoute in 1911 [Sho11]. The *permutahedron* $\Pi_n \in \mathbb{R}^n$ is an $(n-1)$-dimensional polytope defined as the convex hull of all vectors that are obtained by permuting the coordinates of $\mathbf{v} = (1, 2, \dots, n) \in \mathbb{R}^n$. Each of its vertices $(x_1, x_2, \dots, x_n)$ can be identified as a permutation in $\mathfrak{S}_n$ via the map $x_i \mapsto i$. Two vertices are adjacent if and only if their corresponding permutations differ by an adjacent transposition.

We can use a Schlegel diagram to visualize $\mathcal{PF}_n$ in a lower-dimensional space. Loosely speaking, a *Schlegel diagram* projects a polytope in $\mathbb{R}^n$ into $\mathbb{R}^{n-1}$ by projecting from a point just outside the polytope, preserving the structure of each face (A formal definition of Schlegel diagrams can be found in [Zie95, pg. 133], but for our purposes it suffices to rely on this visual intuition). Figure 14a is the Schlegel diagram of $\mathcal{PF}_3$.

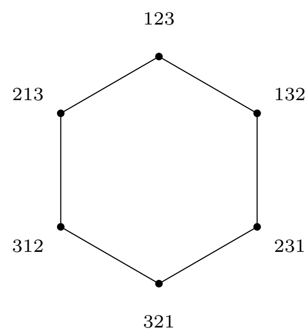


Figure 13. Permutahedron Π_3.

We observe that not all parking functions are vertices of parking function polytopes. For example, the 10 vertices of $\mathcal{PF}_3$ consist of six permutations of $(1, 2, 3)$, three permutations of $(1, 1, 3)$, and $(1, 1, 1)$. The remaining six parking functions of length 3 are permutations of $(1, 1, 2)$ and $(1, 2, 2)$, which either lie on the edges or facets of $\mathcal{PF}_3$, see Figure 14b. For instance, $(1, 1, 2)$ lies on the edge between $(1, 1, 1)$ and $(1, 1, 3)$, forming a linear sequence $(1, 1, 1)$, $(1, 1, 2)$, and $(1, 1, 3)$.

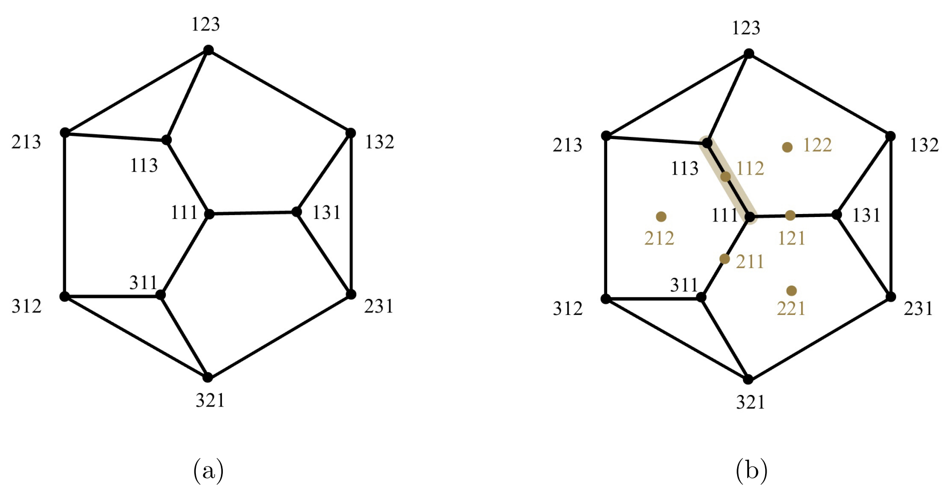


Figure 14. Schlegel diagram of $\mathcal{PF}_3$

These observations lead to two important general principles regarding the vertices of the parking function polytope:

1. If a parking function is not a vertex of the polytope, increasing any entry greater than 1 by 1 yields another parking function (In other words, it is possible to take a "one-step" forward).

2. If a parking function is a vertex of the polytope, then all of its rearrangements are also vertices.

The below proposition by Stanley formalizes these ideas.

PROPOSITION 3.2 ([AW21, pg. 2]). *For each positive integer n, $\mathbf{v} = (v_1, \ldots, v_n)$ is a vertex of $\mathcal{PF}_n$ if and only if it is a rearrangement of*

$$(\underbrace{1, \ldots, 1}_{k \text{ times}}, k+1, k+2, \ldots, n),$$

for some $1 \leq k \leq n$.

Proof. First, for $k \in [n]$, observe that any permutation of the sequence

$$\alpha^{(k)} = (\underbrace{1, \ldots, 1}_{k \text{ times}}, k+1, k+2, \ldots, n)$$

is a parking function of length n. Moreover, increasing any such entry greater than 1 in $\alpha^{(k)}$ by 1 results in a sequence that is no longer a parking function. For example,

$$(\underbrace{1, \ldots, 1}_{k \text{ times}}, k+2, k+2, \ldots, n)$$

is not a parking function.

Suppose $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a parking function that is not a rearrangement of any $\alpha^{(k)}$. Then there exists an index i such that for $\alpha_i > 1$, we increase α_i by 1 yielding another parking function

$$\alpha' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_n).$$

This implies that α lies on the line segment connecting α' and

$$\alpha'' = (\alpha_1, \ldots, \alpha_{i-1}, \alpha_i - 1, \alpha_{i+1}, \ldots, \alpha_n).$$

Therefore, α can be expressed as a convex combination

$$\alpha = \frac{1}{2}\alpha' + \frac{1}{2}\alpha''.$$

Thus, α is not a vertex of $\mathcal{PF}_n$.

Conversely, suppose α is a permutation of $\alpha^{(k)}$ for some k. Suppose for contradiction that α is not a vertex of $\mathcal{PF}_n$. Then it can be written as a convex combination of two other parking functions β, γ, which are vertices of $\mathcal{PF}_n$, say

$$\alpha = \lambda\beta + (1-\lambda)\gamma,$$

for some $0 < \lambda < 1$. This implies that there exists an i such that $\alpha_i \neq \beta_i$ and $\alpha_i \neq \gamma_i$. Without loss of generality, $\beta_i < \alpha_i < \gamma_i$ since $0 < \lambda < 1$. Each entry of a parking function must be a positive integer, so $\alpha_i \geq 2$. However, increasing any entry of α will not result in a parking function, so γ is not a parking function, which is a contradiction. □

To deduce the number of vertices of $\mathcal{PF}_n$, we count the distinct permutations of parking function

$$(\underbrace{1,\ldots,1}_{k \text{ times}}, k+1, k+2, \ldots, n)$$

for each $k \in [n]$. The number of such permutations is given by the multinomial coefficient

$$\binom{n}{k,\ \underbrace{1,\ldots,1}_{n-k \text{ times}}} = \frac{n!}{k!}.$$

Summing over k from 1 to n, the total number of vertices of $\mathcal{PF}_n$ is

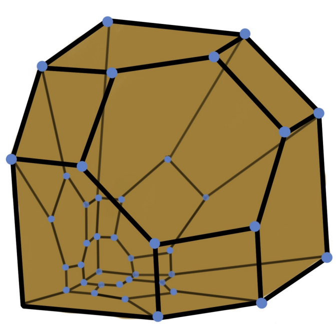
$$n!\left(\frac{1}{1!} + \frac{1}{2!} + \cdots + \frac{1}{n!}\right).$$

If the reader is interested in the number of higher dimensional faces of $\mathcal{PF}_n$, see [AW21, section 2, 3].

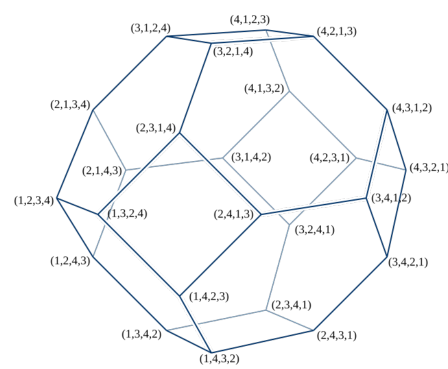
We end this paper by presenting two more examples of parking function polytopes.

Example 3.3. Visualization of $\mathcal{PF}_n$ becomes challenging for $n \geq 5$ (indeed, the Schlegel diagram of $\mathcal{PF}_5$ is 4-dimensional). However, $\mathcal{PF}_4$ is still accessible, and its Schlegel diagram is presented in Figure 15a. Our exposition closely follows that of [MV23, Example 2.3].

$\mathcal{PF}_4$ has only one three-dimensional face, which is the convex hull of 24 vertices, namely, the 24 permutations of $(1, 2, 3, 4)$. The convex hull of these permutations is exactly the permutahedron Π_4 (as shown in Figure 15b). Similarly, we can find 8 two-dimensional faces, each of which are the convex hulls of 6 vertices. These two-dimensional faces are Π_3.



(a) Schlegel diagram of $\mathcal{PF}_4$.



(b) Permutahedron Π_4 [**eppstein2007permutohedron**].

Figure 15

If the reader is interested in a general overview about the face structure of $\mathcal{PF}_n$, see [MV23, Section 2.1].

Example 3.4. *Palindromic parking functions* are parking functions that read the same forwards and backwards. For example, $(1,3,3,1)$ is a palindromic parking function, while $(1,4,2,2)$ is a parking function but not palindromic. Let PPF_n denote the set of palindromic parking functions of length n. Note that parking functions in PPF_n are fixed by the permutation

$$(1,n)(2,n-1)(3,n-2)\cdots \tag{1}$$

since Equation (1) only swaps the ith and $(n-i+1)$th entries, and a palindromic parking function remains unchanged when these entries are exchanged.

To deduce the number of palindromic parking functions, we can use the same reasoning in the proof of Theorem 1.1: We arrange $n+1$ parking spots into a circle and allow $n+1$ to be chosen as a preferred spot. However, this time, only the first $\lfloor (n+1)/2 \rfloor$ cars are able to freely choose any $n+1$ spots to be their preference, since $\lfloor (n+1)/2 \rfloor$ is the number of cycles in Equation (1). For the remaining cars, they need to "duplicate" the preferences to make the list palindromic. Thus, we have

$$|\,\mathrm{PPF}_n\,| = \frac{(n+1)^{\lfloor \frac{n+1}{2} \rfloor}}{n+1} = (n+1)^{\lfloor \frac{n-1}{2} \rfloor}.$$

Define the *palindromic parking function polytope* $\mathcal{PPF}_n$ to be the convex hull of all palindromic parking functions of length n. We describe a bijection between the vertices of $\mathcal{PPF}_{2n}$ and $\mathcal{PF}_n$. Suppose $\mathbf{v} = (v_1, \dots, v_n)$ is a vertex of $\mathcal{PF}_n$ with the form

$$(\underbrace{1,\dots,1}_{k \text{ times}}, k+1, k+2, \dots, n)$$

for some $1 \le k \le n$ and set $u_i = 2v_i - 1$ for each $i \in [n]$. Then $\mathbf{u} = (u_1, \dots, u_n, u_{n+1}, \dots, u_{2n})$ is a vertex of $\mathcal{PPF}_{2n}$. Hence, the palindromic parking function $\mathbf{u}$ of length $2n$ is the vertex of $\mathcal{PPF}_{2n}$ if and only if its first n entries are a rearrangement of

$$(\underbrace{1,\dots,1}_{k \text{ times}}, 2k+1, 2k+3, \dots, 2n-1).$$

Conclusions and Further Readings

The notion of a parking function was first introduced by Konheim and Weiss [KW66] in 1966 as a creative way to describe their work on computer storage. It was quickly observed that the number of parking functions of length n, given by $(n+1)^{n-1}$ [Theorem 1.1], coincides with the number of labeled trees on $n+1$ vertices as given by the famous Cayley formula [**cayley1889theorem**].

This remarkable connection helped establish parking functions as a classical combinatorial object.

Over time, parking functions have found numerous generalizations and applications across various fields including algebra, probability and statistics, representation theory, and geometry. A comprehensive survey that covers the foundational results and developments in the theory of parking functions over the past 30 years is provided in the handbook written by Catherine Yan [**yan2015parking**].

In this paper, we explore parking functions from both an enumerative combinatorial perspective and a geometric perspective. We show that parking functions are not only related to classical Catalan objects such as Dyck paths and noncrossing partitions, but can also be interpreted as well-structured polytopes in Euclidean space. Much of our exposition draw upon the work presented in [Sta97], [AW21], and [MV23], which are excellent sources for further exploration of these two perspectives.

Building on the classical notion, many variants of parking functions have emerged. In particular, we discuss the primitive [Section 1.2] and palindromic parking functions [Example 3.4]. An engaging exposition on different variants of parking functions is available in [Car+20]. Many variants have not yet been studied in depth, and they are great topics for future research.

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