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On Günther's Perturbation Results For Nash's Isometric Embedding Theorems

By Shiv Yajnik

Contents

1. Introduction

Very often, the Riemannian manifolds that we are interested in are explicitly defined as submanifolds of Euclidean space. For example, the n-sphere $\mathbb{S}^n$ can be written as the real solution space of the equation $\sum_{k=1}^{n} x_k^2 = 1$. This equation expresses $\mathbb{S}^n$ as a submanifold of $\mathbb{R}^{n+1}$, and the round metric on this sphere is induced by the Euclidean metric. So, we can easily imagine, for example, what the geodesics may be on these types of submanifolds: They are the curves that are closest to being straight lines in the Euclidean sense. In the case of the sphere, those are the great circles. Similarly, we can also picture what angles might look like without too much difficulty.

But manifolds are not a-priori defined to have a structure whose coordinates are directly inherited from Euclidean space; they are merely required to be *locally Euclidean*–meaning, to consist of Euclidean "patches" that are sewn together. The version of the sphere that we are most familiar with–the one

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with the round metric–is one that can be *isometrically embedded* into three-dimensional Euclidean space. Informally, isometric embeddings are a way to identify a manifold as a subset of Euclidean space in such a way that preserves distances, angles, and curvature intrinsic to the manifold itself. More formally, isometric embeddings preserve the Riemannian metric on the sphere.

We consider two important questions:

- Can any manifold with any metric be isometrically embedded into a Euclidean space, where the embedding is of some desired differentiability class?
- How high is the dimension of the ambient Euclidean space required to be?

This is a very nontrivial problem: For example, there are many metrics on surfaces and higher dimensional spheres which do not admit codimension 1 isometric embeddings, either at all or in the desired differentiability class, as we will see in Sections 2.2 and 3.1.

John Nash proved that any compact Riemannian manifold of dimension n can be smoothly isometrically embedded into $\frac{n(3n+11)}{2}$-dimensional Euclidean space, and that any Riemannian manifold (not necessarily compact) can be smoothly isometrically embedded into $\frac{3}{2}n^3 + 7n^2 + \frac{11}{2}n$-dimensional Euclidean space. Günther, whose methods we will explore in this exposition, improved this bound to

$$\max\left(\frac{n(n+5)}{2}, \frac{n(n+3)}{2} + 5\right)$$

for any Riemannian manifold; this remains the best result for the general case as of 2026.

An isometric embedding is some differentiable map from a manifold M to Euclidean space $\mathbb{R}^N$. As we will see, finding an isometric embedding requires analysis of a system of nonlinear PDE's of first order, which appears in the following form:

$$g_{ij} = \sum_{r=1}^{N} \partial_i u_r \, \partial_j u_r \quad 1 \leq i, j \leq n$$

g_{ij} are known, and $u_1, \ldots, u_N$ are unknown.

In Section 2, we provide a quick introduction to some fundamental notions in Riemannian geometry, including isometric embeddings, and some motivating examples. In Section 3, we build our understanding of the isometric embedding problem as one about PDE's and discuss associated regularity issues, including a certain loss of differentiability which makes this problem especially difficult. The meat of this exposition is contained in Section 4, where we will study the regularity of solutions to the above system of PDE's–in particular, we will examine Matthias Günther's solution to the loss of differentiability problem, which elegantly demonstrates that there exist C^s solutions u, given that g_{ij} is C^s-differentiable for $1 \leq s \leq \infty$. Then, I include a summary of how this perturbation result is applied to prove Günther's version of the isometric embedding theorem.[1] We will conclude with a summary of consequences of the Nash embedding problem, results of isometric embeddings in more specific cases, and a short introduction to global embedding problems in other areas of differential geometry in Sections 5 and 6.

Prerequisites. Sections 2, 3, and 4 assume linear algebra, some real analysis (up to the Arzela–Ascoli theorem) and basic knowledge of manifolds, for example Chapters 1-3, Chapter 5, and Appendices A and B of [57]. Sections 5 and 6 may require some more familiarity with geometry to fully grasp, but I have included citations to literature that more thoroughly discuss many of the areas mentioned.

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2. **Riemannian Geometry and Isometric Embeddings**

2.1. *An Overview of Fundamentals from Riemannian Geometry.* As in [57], I assume that our manifolds are Hausdorff, second countable, and smooth.

We introduce the following notation:

- The space of smooth functions is denoted as $C^\infty(M)$.

[1] Two other useful sources on Nash's and Günther's work on the isometric embedding problem are [2] by Ben Andrews and [38] by Siyuan Lu.

- The tangent space at a point $p \in M$ is denoted T_pM, and the tangent bundle is denoted TM.
- The cotangent space at a point $p \in M$ is denoted T_p^*M, and the cotangent bundle is denoted T^*M.
- For a vector bundle E and for $\lambda > 0$, $C^{k,\lambda}(M, E)$, denotes the space of $C^{k,\lambda}$ sections of E, and $C^\infty(M, E)$ denotes the smoooth sections of E.[2]

In this section, we provide a brief overview of relevant concepts from Riemannian geometry.

First, we review tensor products:

Definition 2.1. *For any commutative ring R, the tensor product $A \otimes B$ of (left) R-modules A and B contains elements $a \otimes b$ with $a \in A$ and $b \in B$ that obey the following relations*[3]:

$$\begin{aligned} ra \otimes b = r(a \otimes b) = (a \otimes rb) \\ (a_1 + a_2) \otimes b = a_1 \otimes b + a_2 \otimes b \\ a \otimes (b_1 + b_2) = a \otimes b_1 + a \otimes b_2 \end{aligned}$$

The spaces $C^\infty(M, TM)$ and $C^\infty(M, T^*M)$ of vector fields and differential 1-forms, respectively, are both $C^\infty(M)$-modules, so we can take tensor products of them. Conveniently, elements of $C^\infty(M, T^*M) \otimes C^\infty(M, T^*M)$, of the form $\omega \otimes \tau$ with $\omega, \tau \in C^\infty(M, T^*M)$, can be expressed at every point $p \in M$ by the $\mathbb{R}$-bilinear map

$$\begin{aligned} (\omega \otimes \tau)_p : T_pM \otimes T_pM \to \mathbb{R} \\ (\omega \otimes \tau)_p(\mathbf{u} \otimes \mathbf{v}) = \omega(\mathbf{u})\omega(\mathbf{v}) \end{aligned}$$

The above corresponds to the definition of 2-tensors given in [57]. Letting p vary across M, we observe $(\omega \otimes \tau)(U \otimes V) = \omega(U)\omega(V) \in C^\infty(M)$ (with $U, V \in C^\infty(M, TM)$). We can repeatedly take tensor products as above to get k-tensors as described in [57].

Definition 2.2. *Define the vector bundle*

$$T^r_s M := TM^{\otimes r} \otimes T^*M^{\otimes s}$$

[2]For further discussion on the meaning of $C^{k,\lambda}$ (Hölder differentiability), see the beginning of Section 3.3.

[3]For more details, in [3], see the beginning of Chapter 1 for definition of commutative rings and Chapter 2 for the definition of modules and tensor products of modules. For the sake of intuition, one can think of a module as being similar to a vector space, but with "scalar multiplication" by elements of a commutative ring such as $C^\infty(M)$ instead of elements of a field such as $\mathbb{R}$.

whose space of sections is[4]

$$C^\infty(M,TM)^{\otimes r} \otimes C^\infty(M,T^*M)^{\otimes s}$$

consisting of (r,s)*-tensors.*

DEFINITION 2.3. *A Riemannian metric* $g : M \to T^*M^{\otimes 2}$ *is a symmetric* $(0,2)$*-tensor such that* $g(p)$ *defines an inner product on the tangent space* T_pM*. A Riemannian manifold* (M,g) *is a manifold* M *equipped with a Riemannian metric* g*.*

The bundle of symmetric $(0,2)$-tensors is denoted $\mathrm{Sym}^2(T^*X)$. The associated space of tensors is $C^\infty(M)$-spanned by elements of the form

$$\mathrm{d}x_i \mathrm{d}x_j = \frac{1}{2}(\mathrm{d}x_i \otimes \mathrm{d}x_j + \mathrm{d}x_j \otimes \mathrm{d}x_i)$$

We will write $\mathrm{d}x^2$ as shorthand for $\mathrm{d}x\mathrm{d}x$.

Any smooth manifold admits a Riemannian metric. $g_{ij} = g(\partial_i, \partial_j)$[5] are smooth functions such that at every point, the matrix with entries $g_{ij}(p)$ is a positive-definite symmetric matrix. We write g^{ij} to denote the entries of the inverse matrix.

We measure rate of change in vector fields with the following device, defined by the Riemannian metric.

DEFINITION 2.4. *Given a Riemannian manifold* (M,g)*, its Levi–Civita connection is the unique map* $\nabla : C^\infty(M,TM) \times C^\infty(M,TM) \to C^\infty(M,TM)$ *which satisfies*:

- *(i)* $C^\infty(M)$*-linearity in the first argument*: $\nabla_{fX+gY}Z = f\nabla_X Z + g\nabla_Y Z$ *for* $X,Y,Z \in C^\infty(M,TM)$ *and* $f,g \in C^\infty(M,TM)$
- *(ii)* $\mathbb{R}$*-linearity in the second argument*: $\nabla_Z(\alpha X + \beta Y) = \alpha\nabla_Z X + \beta\nabla_Z Y$ *for* $X,Y,Z \in C^\infty(M,TM)$ *and* $\alpha,\beta \in \mathbb{R}$
- *(iii) Leibniz rule*: $\nabla_X fY = X(f)Y + f\nabla_X Y$
- *(iv) Torsion freeness*: $\nabla_X Y - \nabla_Y X = [X,Y]$
- *(v) Metric compatibility*: $Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y)$

The Levi–Civita connection is uniquely defined by the Koszul formula:

(1)

$$2g(\nabla_X Y, Z) = X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) - g([X,Z],Y) - g([Y,Z],X) + g([X,Y],Z)$$

[4] $A^{\otimes r}$ is shorthand for r tensor products of A.

[5] $\partial_i = \frac{\partial}{\partial x_i}$

We write ∇_i to represent ∇_{∂_i}. The covariant derivatives ∇_i do not commute in general; in fact, the commutator of two covariant derivatives is a curvature term.

We have that[6] $\nabla_i \partial_j = \Gamma^l_{ij}\partial_l$ for some $\Gamma^l_{ij} \in C^\infty(M)$. Γ^l_{ij} are called the Christoffel symbols, and they can be computed from the metric by the following formula:

$$\Gamma^l_{ij} = \frac{1}{2}\sum_{k=1}^{n} g^{lk}\left(\partial_j g_{ik} + \partial_i g_{kj} - \partial_k g_{ij}\right) \tag{2}$$

Below are definitions for Riemannian and Ricci curvature, respectively, which will appear in the paper:

Definition 2.5. *Riemannian curvature R is defined by*

$$R(X,Y)Z = [\nabla_X, \nabla_Y]Z - \nabla_{[X,Y]}Z$$

The above definition expresses Riemannian curvature as a $(1,3)$ tensor. We can also define Riemannian curvature as a $(0,4)$ tensor given by

$$R(W,X,Y,Z) = g(R(W,X)Y,Z)$$

Definition 2.6. *Ricci curvature is the bilinear form Q given by*[7]

$$\mathrm{Ric}(p)(\mathbf{x},\mathbf{y}) = \mathrm{tr}\left(\mathbf{v} \mapsto R(\mathbf{x},\mathbf{v})\mathbf{y}\right) = \sum_{k=1}^{n} R(p)(\mathbf{e}_k,\mathbf{x},\mathbf{y},\mathbf{e}_k)$$

where $\{\mathbf{e}_1,\ldots,\mathbf{e}_n\}$ forms an orthonormal basis for T_pM.

In terms of local coordinates, we have

$$R(\partial_i,\partial_j)\partial_k = R_{ijk}{}^m\partial_m = [\nabla_i,\nabla_j]\partial_k \qquad R(\partial_i,\partial_j,\partial_k,\partial_l) = R_{ijkl} = R_{ijk}{}^m g_{ml}$$

$$\mathrm{Ric}(\partial_i,\partial_j) = R_{ij} = R_{kij}{}^k = R_{kijl}g^{kl}$$

Notice how Riemannian curvature has four indices, and Ricci curvature comes from multiplying by metric tensor elements and summing over pairs of indices so that Ricci curvature has only two indices. This is an example of *contraction*–that is, the Ricci curvature tensor is a contraction of the Riemannian curvature tensor.

For $X \in C^\infty(M,TM)$, let the $\mathbb{R}$-linear map $\nabla_X : C^\infty(M,TM) \to C^\infty(M,TM)$ be given by the Levi–Civita connection. ∇_X extends to a unique $\mathbb{R}$-linear map $C^\infty(M,T^r_sM) \to C^\infty(M,T^r_sM)$ on (r,s)-tensors satisfying

[6]We invoke the Einstein summation convention here.

[7]We use the Einstein summation convention here.

- $\nabla_X(c(S)) = c(\nabla_X(S))$ for any tensor S and any contraction c.
- Leibniz rule: $\nabla_X(S \otimes T) = (\nabla_X S) \otimes T + S \otimes (\nabla_X T)$ for any tensors S, T.

For $(0, s)$ tensors, letting $Y_1, \ldots, Y_s \in C^\infty(M, TM)$, we have

$$\nabla_X T(Y_1, \ldots, Y_s) = X(T(Y_1, \ldots, Y_s)) - \sum_{i=1}^{s} T(Y_1, \ldots, Y_{i-1}, \nabla_X Y_i, Y_{i+1}, \ldots, Y_s)$$

Finally, we define isometric embeddings, which are the subject of this paper:

Definition 2.7. *Let (M_1, g_1) and (M_2, g_2) be two manifolds with $\dim M_2 \geq \dim M_1$. A smooth function $u : M_1 \to M_2$ is an embedding if the Jacobian of u is of full rank (making u an immersion) and if u is homeomorphic onto its image. This embedding is isometric if $u^* g_2 = g_1$.*

We are interested in finding isometric embeddings of Riemannian manifolds (M, g) into $(\mathbb{R}^N, g_E)$ where g_E is the standard Euclidan metric

$$g_E = \sum_{k=1}^{N} \mathrm{d}x_k^2$$

If u is an isometric embedding of M into $\mathbb{R}^N$, then $u(M)$ is a submanifold of $\mathbb{R}^N$ with the same geodesics and curvature as that of M, so in essence, one is placing M inside of Euclidean space without any deformation or damage to the original Riemannian structure of M.

2.2. *Examples I: Metrics on Spheres.* To get a feel for what manifolds with a given Riemannian structure could look like, we take a short detour to list a few examples of Riemannian metrics and the resulting curvature. These examples are of different metrics on the same types of manifolds: spheres. We illustrate a few of the ways that differences between these structures manifest; in particular, we will see that while one metric may isometrically embed into an Euclidean space of a given dimension N, there may be plenty of other metrics that do not.

Below, we define two more notions of curvature that may be more readily visualizable than the original Riemannian curvature tensor.

Definition 2.8. *Let (M, g) be a Riemannian manifold, R the curvature tensor from the Levi-Civita connection, and σ a 2 dimensional subspace of*

T_pM. We define the sectional curvature of σ to be

$$K(\sigma,p) := \frac{R(p)(\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_2,\mathbf{e}_1)}{\langle \mathbf{e}_1,\mathbf{e}_1\rangle\langle \mathbf{e}_2,\mathbf{e}_2\rangle - \langle \mathbf{e}_1,\mathbf{e}_2\rangle^2}$$

Where $\{\mathbf{e}_1,\mathbf{e}_2\}$ is a basis of σ and $\langle\cdot,\cdot\rangle = g(p)(\cdot,\cdot)$.

One can show that the sectional curvature determines the Riemannian curvature tensor.

Definition 2.9. *Where* Ric *is the Ricci curvature tensor, we define scalar curvature to be the smooth function given by*

$$S(p) := \frac{1}{n(n-1)}\sum_{l=1}^{n} \mathrm{Ric}(p)(\mathbf{e}_l,\mathbf{e}_l)$$

such that $\{\mathbf{e}_1,\dots,\mathbf{e}_n\}$ is an orthonormal basis.[8]

Scalar curvature contains less information but is often simpler to compute and provides perhaps the most friendly measure for how bent a space is around a point.

Example 2.10 (The Round Metric). The standard round metric g_{can} on the sphere $\mathbb{S}^n$ can be expressed as the pullback of the metric g_E on $\mathbb{R}^{n+1}$ under the inclusion map $\iota : \mathbb{S}^n \hookrightarrow \mathbb{R}^{n+1}$ when $\mathbb{S}^n$ is written as the level set $\sum_{k=1}^n x_k^2 = 1$. For example, on $\mathbb{S}^2$, in local coordinates on the northern hemisphere (i.e. $z > 0$), the metric can be written as

$$g_{\mathrm{can}} = \left(1 + \frac{x^2}{1-x^2-y^2}\right)\mathrm{d}x^2 + \left(1 + \frac{y^2}{1-x^2-y^2}\right)\mathrm{d}y^2 - \left(\frac{2xy}{1-x^2-y^2}\right)\mathrm{d}x\mathrm{d}y$$

There is a much nicer expression for g_{can} in spherical coordinates (r,θ_1,θ_2). If we choose the neighborhood of the sphere where $\theta_1 \in (0,\pi)$ and $\theta_2 \in (0,2\pi)$, we can express the metric on the corresponding ambient neighborhood $\mathbb{R}\times(0,\pi)\times(0,2\pi) \subseteq \mathbb{R}^3$ in spherical coordinates by taking the pullback of $\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ under the coordinate transformation:

$$(r,\theta_1,\theta_2) \mapsto (r\cos\theta_1, r\sin\theta_1\cos\theta_2, r\sin\theta_1\sin\theta_2)$$

We get

$$g_E = \mathrm{d}r^2 + r^2\mathrm{d}\theta_1^2 + r^2\sin^2\theta_1\mathrm{d}\theta_2^2$$

[8] In some other conventions, the definition of scalar curvature leaves out the factor of $\frac{1}{n(n-1)}$.

Since the sphere is given by just $r = 1$, pulling back g_E via the inclusion map $\mathbb{S}^2 \hookrightarrow \mathbb{R}^3 \setminus \{0\}$ reveals that the metric on the sphere can be written as

$$g_{\text{can}} = \mathrm{d}\theta_1^2 + \sin^2\theta_1 \mathrm{d}\theta_2^2$$

Using (2) again, we can compute the Christoffel symbols: $\Gamma_{12}^2 = \Gamma_{21}^2 = \cot\theta_1$, $\Gamma_{22}^1 = -\sin\theta_1\cos\theta_1$, and the rest are 0. So we have

$$\begin{aligned}
\nabla_1\partial_1 &= 0\\
\nabla_2\partial_1 &= \nabla_1\partial_2 = \cot\theta_1\partial_2\\
\nabla_2\partial_2 &= -\sin\theta_1\cos\theta_1\partial_1
\end{aligned}$$

With the orthonormal basis $\mathbf{e}_1 = \partial_1$ and $\mathbf{e}_2 = \frac{\partial_2}{\sin\theta_1}$, we compute

$$R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1) = \frac{g_{\text{can}}(\nabla_1\nabla_2\partial_2 - \nabla_2\nabla_1\partial_2,\ \partial_1)}{\sin^2\theta_1} = -\frac{g_{\text{can}}(\nabla_1(\sin\theta_1\cos\theta_1\partial_1) - \nabla_2(\cot\theta_1\partial_2),\ \partial_1)}{\sin^2\theta_1} = 1$$

We get that the sectional curvature K and the scalar curvature S are:

$$\begin{aligned}
K &= \frac{R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1)}{\langle\mathbf{e}_1, \mathbf{e}_1\rangle\langle\mathbf{e}_2, \mathbf{e}_2\rangle - \langle\mathbf{e}_1, \mathbf{e}_2\rangle^2} = 1\\
S &= \frac{1}{2}\left(R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1) + R(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_2, \mathbf{e}_1)\right) = 1
\end{aligned}$$

Hence, the 2-sphere with round metric has constant sectional curvature 1, as does every n-sphere with the round metric. Round spheres also have scalar curvature 1, which is to be expected: scalar curvature here will agree with sectional curvature if the sectional curvature is constant.

Since the round metric on $\mathbb{S}^n$ is the pullback of the standard metric on $\mathbb{R}^{n+1}$ along inclusion, we have automatically that $(\mathbb{S}^n, g_{\text{can}})$ smoothly isometrically embeds into $(\mathbb{R}^{n+1}, g_E)$.

Example 2.11 (Berger Spheres). The *quaternions* are a unital associative $\mathbb{R}$-algebra of elements of the form $z = v + w\mathbf{i} + x\mathbf{j} + y\mathbf{k}$ with $w, x, y, z \in \mathbb{R}$ and relations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{ijk} = -1$. These can be thought of as an extension of the complex numbers. We define $\bar{z} = v - w\mathbf{i} - x\mathbf{j} - y\mathbf{k}$.

$\mathbb{S}^3$ can be expressed as the set of unit quaternions, given by $|z| = z\bar{z} = 1$. The unit quaternions form a noncommutative *Lie group*. In this case, this means that the unit quaternions have a manifold structure combined with an associative arithmetic operation–quaternion multiplication–with identity element 1 and a multiplicative inverse for every element. On $\mathbb{S}^3$, multiplication $\cdot : \mathbb{S}^3 \times \mathbb{S}^3 \to \mathbb{S}^3$ is smooth, and so is the associated inverse map $()^{-1} : \mathbb{S}^3 \to \mathbb{S}^3$. Quaternion multiplication, unlike multiplication of complex numbers, is *noncommutative*.

On a Lie group, every tangent space is canonically isomorphic to the tangent space at the identity by pushfoward along left multiplication by group elements; the tangent space at the identity is referred to as the *Lie algebra*.

The Lie algebra $T_1\mathbb{S}^3$ of $\mathbb{S}^3$ consists of imaginary quaternions, of the form $w\mathbf{i}+x\mathbf{j}+y\mathbf{k}$. This Lie algebra is generated by the basis $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ satisfying

$$[\mathbf{i},\mathbf{j}] = 2\mathbf{k}, \quad [\mathbf{j},\mathbf{k}] = 2\mathbf{i}, \quad [\mathbf{k},\mathbf{i}] = 2\mathbf{j} \tag{3}$$

Because of the canonical isomorphism of tangent spaces at every point, we can define a metric on $\mathbb{S}^3$ by specifying an inner product on the Lie algebra and defining the inner product at other points simply to be the pushforward of the inner product along left multiplication of group elements.

We define a metric in this way by declaring an orthonormal basis to be $\varepsilon\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ for a constant $\varepsilon \in \mathbb{R}^+$. This defines a family of metrics g^ε on $\mathbb{S}^3$; $(\mathbb{S}^3, g^\varepsilon)$ are known as the Berger Spheres, after Marcel Berger.[9] g^1 is the round metric. Using the Koszul formula (1), we now compute the covariant derivatives and the curvature tensor. Let ∇^ε denote the Levi–Civita connection of g^ε.

$$g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{i},\mathbf{i}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{i},\mathbf{j}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{i},\mathbf{k}) = 0 \Longrightarrow \nabla^\varepsilon_{\mathbf{i}}\mathbf{i} = 0$$

$$g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{j},\mathbf{i}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{j},\mathbf{j}) = 0,\ g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{j},\mathbf{k}) = 2-\frac{1}{\varepsilon^2} \Longrightarrow \nabla^\varepsilon_{\mathbf{i}}\mathbf{j} = \left(2-\frac{1}{\varepsilon^2}\right)\mathbf{k} \Longrightarrow \nabla^\varepsilon_{\mathbf{j}}\mathbf{i} = -\frac{1}{\varepsilon^2}\mathbf{k}$$

$$g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{k},\mathbf{i}) = 0,\ g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{k},\mathbf{j}) = -2+\frac{1}{\varepsilon^2},\ g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\mathbf{k},\mathbf{k}) = 0 \Longrightarrow \nabla^\varepsilon_{\mathbf{i}}\mathbf{k} = \left(-2+\frac{1}{\varepsilon^2}\right)\mathbf{j} \Longrightarrow \nabla^\varepsilon_{\mathbf{k}}\mathbf{i} = \frac{1}{\varepsilon^2}\mathbf{j}$$

$$g^\varepsilon(\nabla^\varepsilon_{\mathbf{j}}\mathbf{j},\mathbf{i}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{j}}\mathbf{j},\mathbf{j}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{j}}\mathbf{j},\mathbf{k}) = 0 \Longrightarrow \nabla^\varepsilon_{\mathbf{j}}\mathbf{j} = 0$$

$$g^\varepsilon(\nabla^\varepsilon_{\mathbf{j}}\mathbf{k},\mathbf{i}) = \frac{1}{\varepsilon^2},\ g^\varepsilon(\nabla^\varepsilon_{\mathbf{j}}\mathbf{k},\mathbf{j}) = 0,\ g^\varepsilon(\nabla^\varepsilon_{\mathbf{j}}\mathbf{k},\mathbf{k}) = 0 \Longrightarrow \nabla^\varepsilon_{\mathbf{j}}\mathbf{k} = \mathbf{i} \Longrightarrow \nabla^\varepsilon_{\mathbf{k}}\mathbf{j} = -\mathbf{i}$$

$$g^\varepsilon(\nabla^\varepsilon_{\mathbf{k}}\mathbf{k},\mathbf{i}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{k}}\mathbf{k},\mathbf{j}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{k}}\mathbf{k},\mathbf{k}) = 0 \Longrightarrow \nabla^\varepsilon_{\mathbf{k}}\mathbf{k} = 0$$

With the orthonormal basis $\mathbf{e}_1 = \varepsilon\mathbf{i}$, $\mathbf{e}_2 = \mathbf{j}$, and $\mathbf{e}_3 = \mathbf{k}$, we find

$$R(\varepsilon\mathbf{i},\mathbf{j},\mathbf{j},\varepsilon\mathbf{i}) = \varepsilon^2 g^\varepsilon(\nabla^\varepsilon_{\mathbf{i}}\nabla^\varepsilon_{\mathbf{j}}\mathbf{j} - \nabla^\varepsilon_{\mathbf{j}}\nabla^\varepsilon_{\mathbf{i}}\mathbf{j} - \nabla^\varepsilon_{[\mathbf{i},\mathbf{j}]}\mathbf{j},\ \mathbf{i}) = \frac{1}{\varepsilon^2}$$

$$R(\varepsilon\mathbf{i},\mathbf{k},\mathbf{k},\varepsilon\mathbf{i}) = \varepsilon^2 g^\varepsilon\left(\nabla^\varepsilon_{\mathbf{i}}\nabla^\varepsilon_{\mathbf{k}}\mathbf{k} - \nabla^\varepsilon_{\mathbf{k}}\nabla^\varepsilon_{\mathbf{i}}\mathbf{k} - \nabla^\varepsilon_{[\mathbf{i},\mathbf{k}]}\mathbf{k},\ \mathbf{i}\right) = \frac{1}{\varepsilon^2}$$

$$R(\mathbf{j},\mathbf{k},\mathbf{k},\mathbf{j}) = g^\varepsilon(\nabla^\varepsilon_{\mathbf{j}}\nabla^\varepsilon_{\mathbf{k}}\mathbf{k} - \nabla^\varepsilon_{\mathbf{k}}\nabla^\varepsilon_{\mathbf{j}}\mathbf{k} - \nabla^\varepsilon_{[\mathbf{j},\mathbf{k}]}\mathbf{k},\ \mathbf{j}) = 4-\frac{3}{\varepsilon^2}$$

[9]Berger spheres are included as Example 1.3.5. and discussed in Section 4.4.3 of [47]. Some of the calculations here can be found there as well.

We find that the sectional curvatures $K(\sigma)$ at $\sigma = \mathrm{Span}_{\mathbb{R}}(\mathbf{i},\mathbf{j}),\ \mathrm{Span}_{\mathbb{R}}(\mathbf{i},\mathbf{k}),\ \mathrm{Span}_{\mathbb{R}}(\mathrm{j},\mathbf{k})$ and the scalar curvature S is given by:

$$K(\mathrm{Span}_{\mathbb{R}}(\mathbf{i},\mathbf{j})) = \frac{R(\varepsilon\mathbf{i},\mathbf{j},\mathbf{j},\varepsilon\mathbf{i})}{\langle\varepsilon\mathbf{i},\varepsilon\mathbf{i}\rangle\langle\mathbf{j},\mathbf{j}\rangle - \langle\varepsilon\mathbf{i},\mathbf{j}\rangle^2} = \frac{1}{\varepsilon^2}$$
$$K(\mathrm{Span}_{\mathbb{R}}(\mathbf{i},\mathbf{k})) = \frac{R(\varepsilon\mathbf{i},\mathbf{k},\mathbf{k},\varepsilon\mathbf{i})}{\langle\varepsilon\mathbf{i},\varepsilon\mathbf{i}\rangle\langle\mathbf{k},\mathbf{k}\rangle - \langle\varepsilon\mathbf{i},\mathbf{k}\rangle^2} = \frac{1}{\varepsilon^2}$$
$$K(\mathrm{Span}_{\mathbb{R}}(\mathbf{j},\mathbf{k})) = \frac{R(\mathbf{j},\mathbf{k},\mathbf{k},\mathbf{j})}{\langle\mathbf{j},\mathbf{j}\rangle\langle\mathbf{k},\mathbf{k}\rangle - \langle\mathbf{j},\mathbf{k}\rangle^2} = 4 - \frac{3}{\varepsilon^2}$$

$$S = \frac{1}{6}(2R(\varepsilon\mathbf{i},\mathbf{j},\mathbf{j},\varepsilon\mathbf{i}) + 2R(\varepsilon\mathbf{i},\mathbf{k},\mathbf{k},\varepsilon\mathbf{i} + 2R(\mathbf{j},\mathbf{k},\mathbf{k},\mathbf{j})) = \frac{1}{6}\left(8 - \frac{2}{\varepsilon^2}\right)$$

The above curvature data shows that the metrics g^ε are very different from the round metric g^1:

- The scalar curvature can be zero or negative depending on the value of ε.
- The sectional curvature $K(p,\sigma)$ is non-constant with respect to σ if $\varepsilon \neq 1$.

Also unlike the round metric, infinitely many of the metrics g^ε for $\varepsilon > 0$ do not admit *conformal* embeddings,[10] let alone isometric embeddings, by Corollary 2.2 of Chiakuei Peng and Zizhou Tang in [46]; they prove this by an argument involving the Chern–Simons 3-form.

3. **Regularity Issues, Perturbations, and Hölder Norms**

Recall that given a metric g on M, we are to find an embedding $u : M \to \mathbb{R}^N$ such that $u^*g_E = g$. Equivalently, we may look at this problem in the following way: Any immersion $u = (u_1, \ldots, u_N) : M \to \mathbb{R}^N$ induces a tensor $g_u = u^*g_E$ given by

$$(g_u)_{ij} = \partial_i u \cdot \partial_j u = \sum_{r=1}^{N} \partial_i u_r \, \partial_j u_r$$

Proving the isometric embedding theorem amounts to finding u such that

$$g_u = g \tag{4}$$

We will see in this section that our problem is one in analysis as well as geometry.

The following is the precise version of Nash's Isometric Embedding Theorems for compact Riemannian manifolds:

[10] Isometric embeddings have to be conformal embeddings, but conformal embeddings are weaker than isometric embeddings, as they preserve only angles, not necessarily distance.

Theorem 3.1 ([42] Theorem 2). *Any compact Riemannian n-manifold with a C^s metric has a C^s isometric embedding in $\mathbb{R}^{n(3n+11)/2}$ for $3 \leq s \leq \infty$.*

Theorem 3.2 ([42] Theorem 3). *Any Riemannian manifold with a C^s metric has a C^s embedding in $\mathbb{R}^{\frac{3}{2}n^3+7n^2+\frac{11}{2}n}$ for $3 \leq s \leq \infty$.*

Nash's isometric embedding theorems are not only geometric results: They are also *regularity*[11] results in the context of PDE's, and (4) is a nonlinear system of PDE's. In this paper, we will explore Günther's version of this result, which is similarly demanding in terms of regularity.

3.1. *Examples II*: *C^2 vs C^1 Embeddings on Spheres and Surfaces.* If the dimension N of the ambient Euclidean space is too low, a C^s metric may admit an isometric embedding of a lower differentiability class into $\mathbb{R}^N$, but not necessarily a C^s embedding. In particular, we will see that C^2 is a substantially more rigid differentiability class than C^1.

We start with some definitions and a very interesting result about C^1-isometric embeddings by Nash and Nicolaas Kuiper:

Definition 3.3. *A strictly short map f of Riemannian manifolds (M_1, g_1), (M_2, g_2) satisfies $\|\mathrm{d}f(\mathbf{v})\|_{g_2(f(p))} < \|\mathbf{v}\|_{g_1(p)}$ for all $p \in M_1$, $\mathbf{v} \in T_pM_1$.*[12]

The most general definition of a strictly short map is on metric spaces (X_1, d_1), and (X_2, d_2), where strict shortness of a map $f : X_1 \to X_2$ is defined by the property $d_1(x, y) > d_2(f(x), f(y))$ for all $x, y \in X_1$. On a Riemannian manifold, there is a notion of distance:

Definition 3.4. *Let (M, g) be a Riemannian manifold. Let $\gamma : [0, 1] \to M$ be a path between two points p_1 and p_2 (i.e. $\gamma(0) = p_1$, $\gamma(1) = p_2$). The length $\ell(\gamma)$ of γ is given by*

$$\ell(\gamma) = \|\gamma'(t)\|_{g(\gamma(t))}$$

The distance $\mathrm{dist}_g(p_1, p_2)$ between p_1 and p_2 with respect to the metric g is defined to be

$$\mathrm{dist}_g(p_1, p_2) = \inf\{\ell(\gamma) : \gamma \textit{ is piecewise smooth}, \ \gamma(0) = p_1, \ \gamma(1) = p_2\}$$

[11] *Regularity* is an umbrella term we use to refer to "niceness" of solutions to PDE's; examples of "niceness include C^s differentiability and L_p integrability.

[12] $\|\cdot\|_{g(p)}$ and $\|\cdot\|_{g_i(p)}$ are the metric-induced norms on M_i given by $\|\mathbf{v}\|_{g_i(p)} = \sqrt{g_i(p)(\mathbf{v}, \mathbf{v})}$.

Hence, shortness of maps on Riemannian manifolds holds only if the map is short on the underlying metric spaces with the distance metrics induced by the Riemannian metric tensors.

THEOREM 3.5 (Nash–Kuiper, [19] Theorem 21.2.1). *For two Riemannian manifolds* (M_1, g_1) *and* (M_2, g_2)[13] *with* $\dim M_i = N_i$, *if* $N_1 < N_2$, *then any strictly short immersion (not necessarily isometric) can be* C^0*-approximated by a uniformly convergent sequence of isometric* C^1*-embeddings.*

This means that the C^1 class of isometric embeddings is *very* flexible and that a large class of manifolds admit C^1-isometric embeddings of codimension just 1, for example. But many of these manifolds will not admit isometric embeddings of higher differentiability class.

Example 3.6. (Berger Sphere, revisited) In the previous section, we introduced the Berger metrics on the 3-sphere and mentioned that a corollary of Peng–Tang's result in [46] is that infinitely many of these metrics admit no conformal C^2-embeddings into $\mathbb{R}^4$. But, we can easily find a short immersion of the Berger sphere into $\mathbb{R}^4$: Given a Berger metric g^ε on the sphere with $\varepsilon \neq 1$, we can use inclusion of $\mathbb{S}^3 \hookrightarrow \mathbb{R}^4$ as the unit quaternions, then scale this inclusion so that the radius of the immersed sphere in $\mathbb{R}^4$ is less than ε. In other words, we define our immersion f by

$$z \overset{f}{\mapsto} rz \quad r < \varepsilon$$

where z is a unit quaternion. By comparing the Berger metrics g^ε with the metric on the immersed sphere, which is rg_{can}, it is simple to check that f is a short immersion. Therefore, we can apply Nash–Kuiper to conclude that there is a C^1-isometric embedding of the 3-sphere with the Berger metric g^ε into $\mathbb{R}^4$ for any constant ε.

We will now show that any closed orientable surface admits metrics with isometric C^1-embeddings into $\mathbb{R}^3$ but no isometric C^2-embeddings into $\mathbb{R}^3$.

[13] Closedness need not be assumed here.

It is known by the uniformization theorem[14] that every closed orientable surface admits a metric with Gaussian curvature 1, 0, or -1; in particular, the sphere admits a metric with positive curvature 1, the torus admits a flat metric, and the g-holed tori for $g \geq 2$ admit hyperbolic metrics (i.e. curvature -1).[15]

Example 3.7 (g-Holed Tori). To see that an isometric C^1-embedding exists for the flat metric on the torus and the hyperbolic metric on the g-holed tori for $g \geq 2$, we can consider *normal coordinates*; the following is an example of what we mean.[16]

Take a small disk from the flat torus, a small disk from the hyperbolic surface, and a small diskfrom the corresponding embedded surface. Hyperbolic surfaces are locally modeled on the *Poincaré disk*[17] D_H with hyperbolic metric

$$g_H = \frac{4\mathrm{d}x^2 + 4\mathrm{d}y^2}{(1-(x^2+y^2))^2}$$

The flat torus is modelled on the flat disk[18] D_E with Euclidean metric

$$g_E = \mathrm{d}x^2 + \mathrm{d}y^2$$

On the embedded surfaces, one can take the coordinates on a small disk D_S to be orthogonal projection $\pi_\perp$ onto the tangent plane so that locally the metric on D_S is of the form

$$g_S = \mathrm{d}(x \circ \pi_\perp)^2 + \mathrm{d}(y \circ \pi_\perp)^2 = \rho_1(x,y)\mathrm{d}x^2 + \rho_2(x,y)\mathrm{d}y^2$$

[14]See Chapter 11 [51] for more on the uniformization theorem, which states that every simply connected Riemann surface is biholomorphic to $\mathbb{S}^2$, the complex plane $\mathbb{C}$, or the upper half plane $\mathbb{H}$, which admit metrics with constant sectional (or Gaussian) curvature 1, 0, and -1, respectively. Tori can be realized as $\mathbb{C}/\Lambda$ where Λ is an integer lattice in $\mathbb{C}$, and g-holed tori for $g \geq 2$ can be realized as the quotient of H by a finite subgroup of $PSL(2,\mathbb{R})$, and these inherit the Riemannian structures (and therefore curvature as well) from their universal covers $\mathbb{C}$ and $\mathbb{H}$.

[15]*Gaussian curvature* is the product of the minimum and maximum possible curvature of curves passing through a point rendered as cross sections of the surface with a plane containing a vector normal to the surface. In two dimensions, it is equal to the sectional curvature.

[16]For a more rigorous explanation of geodesics and normal coordinates, see Sections 14 and 15 in [56].

[17]The Poincaré disk is (biholomorphically) isometric to the upper half plane with metric $\frac{\mathrm{d}x^2+\mathrm{d}y^2}{y^2}$.

[18]Usually, we depict a torus as a square with parallel edges identified, but for the sake of this discussion, we take the local picture to be a disk; one can think of this disk as a subset of the square.

for some smooth functions ρ_i. One can find that curves of minimal distance–or geodesics: On D_H, these are circular arcs perpendicular to the boundary of the disk and diameters, and on D_E, these are lines. The image of D_S through orthogonal projection is D_E, and one can find by staring at this picture that for three points p_1, p_2, and p_3 on the disk from the embedded surface,

$$\text{dist}_{g_S}(p_1, p_2) > \text{dist}_{g_S}(p_1, p_3) \iff \text{dist}_{g_E}(\pi_\perp(p_1), \pi_\perp(p_2)) > \text{dist}_{g_E}(\pi_\perp(p_1), \pi_\perp(p_3))$$

The quantity

$$|\text{dist}_{g_S}(p_1, p_2) - \text{dist}_{g_E}(\pi_\perp(p_1), \pi_\perp(p_2))|$$

has an upper bound for all p_1, p_2 by boundedness of the disk.

After staring some more–in particular, in the case of the hyperbolic surfaces, comparing (D_H, g_H) with (D_E, g_E)–one can conclude that obtaining short immersions of the flat torus and hyperbolic g-holed tori into $\mathbb{R}^3$ is possible if one is able globally apply a shrinking of D_S to the entire embedded surface; we can do this simply by globally shrinking the embedded surfaces in $\mathbb{R}^3$, so Nash–Kuiper applies.

However, these metrics on the g-holed tori admit no C^2-embeddings into $\mathbb{R}^3$ because any smooth hypersurface of $\mathbb{R}^3$ with metric induced by the Euclidean metric must have at least one point with positive Gaussian curvature–in other words, it must be locally strictly convex around at least one point.[19] Therefore, we know automatically that the g-holed tori admit metrics that do not C^2-embed isometrically into $\mathbb{R}^3$ with the Euclidean metric for all $g \geq 1$.

Next is an example of a metric on $\mathbb{S}^2$ admitting no isometric C^2-embedding into $\mathbb{R}^3$.

Example 3.8 (Greene's Sphere, [24]). Consider the stereographic projection of $\mathbb{S}^2 \setminus \{p_S\}$ onto $\mathbb{R}^2$, that takes southern hemisphere $H_2 = \{z < 0\}$ onto the open unit disk on the xy-plane and the equator $Z = \{z = 0\}$ onto the unit circle on the xy-plane. $\mathbb{S}^2 \setminus \{p_S\}$ inherits a flat metric g_1 from $\mathbb{R}^2$.

Then, for $\varepsilon > 0$, define the open annulus $A_\varepsilon \subseteq \mathbb{S}^2 \setminus \{p_S\}$ by

$$A_\varepsilon = \{p \in \mathbb{S}^2 \setminus \{p_S\} : \inf_{q \in E} \text{dist}_{g_1}(q, p) < \varepsilon\}$$

Let $f_\varepsilon : S \setminus \{p_S\}$ be a smooth bump function adjusted so that, where $H_1 = \{z > 0\}$ is the northern hemisphere:

- $f_\varepsilon(p) = 1$ for $p \in H_1 \cup A_{\varepsilon/2}$
- $f_\varepsilon(p) < \varepsilon$ for $p \in \mathbb{S}^2 \setminus (H_1 \cup A_\varepsilon)$

[19] See Exercise 16 in Section 3.3 of [13].

- $0 < f_\varepsilon(p) \leq 1$ for $p \in \mathbb{S}^2$

One can extend $g_1|_{H_1 \cup A_{1/2}}$ to all of $\mathbb{S}^2$ by partition of unity; call this new metric g_2. Then let $g = f_\varepsilon g_2$ for ε small enough so that

$$\sup_{p_1, p_2} \mathrm{dist}_g(p_1, p_2) \leq \frac{1}{2}$$

g here is the C^∞ metric constructed by Robert Greene in [24] that admits no C^2-isometric embedding into $\mathbb{R}^3$ with the Euclidean metric.

One can apply the idea of normal coordinates in a manner similar to that of Example 3.7 to show that one can find a short immersion of the sphere by shrinking the embedded sphere in $\mathbb{R}^3$, so by Nash–Kuiper, the sphere with the above metric g admits an isometric C^1-embedding.

His idea was to deform and flatten the lower hemisphere in such a way that the structures are still smooth, but all of the distances with respect to the sphere's metric between points on the equator shrink–namely, all at most $\frac{1}{2}$.

Since the shortest path between two equatorial points traverses the flat part of the structure on the sphere, it must therefore be true that the distance between these points' images in $\mathbb{R}^3$ must also be at most $\frac{1}{2}$.

However, Greene proved that an obstruction to C^2 isometric embeddings is that there must exist points on the equator whose images are at least $\sqrt{3}$ apart in $\mathbb{R}^3$.

One fascinating project inspired by the Nash–Kuiper theorem is the Hevea Project, which is dedicated to visualizing surfaces with metrics that admit only isometric C^1-embeddings into $\mathbb{R}^3$ based on explicit constructions of the closed forms of such metrics. As of 2026, the project includes renderings of the flat torus, the hyperbolic plane– $\mathbb{R} \times \mathbb{R}^+$ endowed with the metric $\frac{\mathrm{d}x^2 + \mathrm{d}y^2}{y^2}$–and the reduced sphere–an object obtained by using Nash-Kuiper to perform a deformation in the C^1-family approximating the round metric so that we end up with a sphere of arbitrarily small radius but with the geodesic length preserved from the original unit sphere.

3.2. *Linearizing the Problem by Choice of Variations.* Returning to the general isometric embedding problem, we recall that, given a C^s embedding of a Riemannian n-manifold (M, g) into $(\mathbb{R}^N, g_E)$, the aim is to find a C^s

embedding u such that

$$g_{ij} = (g_u)_{ij} = \sum_{r=1}^{N} \partial_i u_r \, \partial_j u_r \quad 1 \leq i, j \leq n$$

The above system of PDE's is very nonlinear; one cannot learn much by trying to solve directly for u. One useful method to get the correct regularity result for the above system is to consider the following *perturbation problem*: If $v = (v_1, \ldots, v_N) : M \to \mathbb{R}^N$ is a variation in the embedding inducing a change–or *perturbation*–f in the metric, we get the following system of first-order PDE:

$$g_{u+v} = g_u + f \iff \sum_{r=1}^{N} \partial_i(u_r + v_r) \, \partial_j(u_r + v_r) = \sum_{r=1}^{N} \partial_i u_r \, \partial_j u_r + f_{ij} \quad 1 \leq i, j \leq n$$

$$(5) \qquad \iff \sum_{r=1}^{N} \partial_i u_r \, \partial_j v_r + \sum_{r=1}^{N} \partial_i v_r \, \partial_j u_r + \sum_{r=1}^{N} \partial_i v_r \, \partial_j v_r = h_{ij} \quad 1 \leq i, j \leq n$$

More precisely, if the perturbation f is C^s, we need to prove that there exists a C^s variation v inducing f.[20]

The system of PDE's (5) is *still* nonlinear. But our redefinition of our problem in terms of perturbations allows us to employ some tricks.

We define the embedding u to be *free* if the vectors

$$\begin{bmatrix} \partial_i u_1 \\ \vdots \\ \partial_i u_N \end{bmatrix} \quad i \in \{1, \ldots, n\} \qquad\qquad \begin{bmatrix} \partial_{ij} u_1 \\ \vdots \\ \partial_{ij} u_N \end{bmatrix} \quad 1 \leq i \leq j \leq n$$

are linearly independent.

Nash simplified this problem by considering *normal* variations: That is, our C^s variation v satisfies

$$\sum_{r=1}^{N} v_r \cdot \partial_j u_r = 0 \qquad 1 \leq j \leq n$$

[20] Technically, as we will see in the statement of Günther's perturbation result (Theorem 4.1, the proof is for *sufficiently small* C^s-differentiable f.

Then, by product rule,

$$\implies \sum_{r=1}^{N} \partial_i v_r \; \partial_j u_r + \sum_{r=1}^{N} v_r \; \partial_{ij} u_r = 0$$

Then, when v is a normal variation, our PDE becomes

$$-2\sum_{r=1}^{N} v_r \; \partial_{ij} u_r + \sum_{r=1}^{N} \partial_i v_r \; \partial_j v_r = f_{ij} \quad 1 \leq i, j \leq n$$

Noting that the second term on the left is quadratic in v (hence for v small, it contributes negligibly), if our perturbation $f = tf_0$, the infinitesimal variation $w = \partial_t v|_{t=0}$ will satisfy

$$\text{(6)} \qquad \sum_{r=1}^{N} w_r \; \partial_{ij} u_r = -\frac{1}{2}(f_0)_{ij}, \quad \sum_{r=1}^{N} w_r \; \partial_j u_r = 0 \quad 1 \leq i, j \leq n$$

This is seen by taking the derivative with respect to t in the PDE and the normal condition on v. u is a known embedding, and the variation w_r is unknown–Nash has *linearized* this problem by constructing a linear system of equations for w_r.

There remains a key issue, however. If one looks carefully at the PDE, one would notice that if v is of class C^s, then $f = g_{u+v} - g_u$ is of class C^{s-1} because of the first order partial derivatives; correspondingly, if h is of class C^{s-1}, then the normal infinitesimal variation w must also be of class C^{s-1}. So in this process, we have lost a degree of differentiability, which poses an obstacle to being able to estimate our Riemannian metric in the class C^s. We refer to this as the "loss of differentiability" problem.

Nash's original solution to this problem emerged in the following way: Viewing $v \mapsto g_{u+v} - g_u$ as a smooth map $C^s(M, NM) \to C^{s-1}(M, \mathrm{Sym}^2(T^*M))$ of Banach spaces,[21] one can think of our objective as attempting to prove a type of inverse function theorem. But our observations show that we cannot use the conventional inverse function theorem from real analysis to return a C^s variation from a C^s perturbation f because the inverse of the derivative of this map, given by the infinitesimal variation w, will not be bounded in C^s.

Nash proved the required inverse function theorem in [42]. Nash's solution to the loss of differentiability problem turns out to be a momentous feat in mathematics; it ultimately allowed Nash to solve the perturbation problem

[21] NM here is the *normal bundle* of M as an embedded submanifold into $\mathbb{R}^N$, defined to be the orthogonal complement of TM in $T\mathbb{R}^N$.

and prove Theorems 3.1 and 3.2. Nash's proof also had implications for other problems in geometry and analysis, which will be briefly discussed in Section 5.

The proof of this inverse function theorem is monstrous. Günther discovered a way to avoid the loss of differentiability, however. His solution will be the subject of Section 4.

3.3. *Hölder Norms.* Before we begin discussing a solution to the general isometric embedding problem, however, we need more definitions: In order to prove obtain a perturbation result with the right regularity, one must provide estimates in the proper C^s-differentiable class. To make estimations precise, it would be especially convenient if the spaces of functions we are working in are complete so that we can use tools such as Arzela–Ascoli to do approximations.

Definition 3.9 (Hölder-differentiablity). *Functions of class $C^{s,\lambda}(D)$ on some domain $D \subset \mathbb{R}$ are those f whose kth derivative must satisfies the following Hölder condition*:

$$\exists C \geq 0 : \forall\, x, y,\ |f(x) - f(y)| \leq C\|x - y\|^{\lambda}$$

This definition generalizes to functions on $\mathbb{R}^n$.

Unlike the space of $C^s = C^{s,0}$ functions, the space of $C^{s,\lambda}$ functions has a complete norm-induced topology for $0 < \lambda < 1$. Let $0 < \lambda < 1$ be fixed. I use the following definition for $C^{s,\lambda}$ norms given by Günther in [28] (generally referred to as Hölder norms):

Definition 3.10. *For a real function $u : B \to \mathbb{R}$ where B is an open m-ball around the origin, and for $0 < \lambda < 1$, we define the $C^{0,\lambda}$ (or C^{λ}) norm $\|\cdot\|_0$ and the $C^{s,\lambda}$ norm $\|\cdot\|_s$ by*

$$\|u\|_0^B := \sup_{x\in B} |u(x)| + \sup_{x,y\in B,\, x\neq y} |x-y|^{-\lambda}|u(x) - u(y)|$$

$$\|u\|_s^B := \|u\|_0^B + \sum_{i_1,\dots,i_s=1}^{m} \|\partial_{i_1\dots i_s} u\|_0^B$$

One simple but important fact that we will use to absorb lower-order norms into higher order norms is that

$$\|u\|_s^B \leq K\|u\|_{s+1}^B \tag{7}$$

Intuitively, this makes sense because $C^{s+1,\lambda}$ is a better regularity than $C^{s,\lambda}$. To show this using the definition, we observe that

$$\begin{aligned}|\partial_{i_1\ldots i_s}u(x)-\partial_{i_1\ldots i_s}u(y)| &\leq \sum_j \left|\int_B \partial_{ji_1\ldots i_s}u(x)-\partial_{ji_1\ldots i_s}u(y)\mathrm{d}x_j\right| \\ &\leq \mathrm{Vol}(B)\sum_j \sup_{x,y\in B,\, x\neq y}|\partial_{ji_1\ldots i_s}u(x)-\partial_{ji_1\ldots i_s}u(y)|\end{aligned}$$

From this, we can conclude that

$$\begin{aligned}&\sum_{i_1,\ldots,i_s=1}^{m}\sup_{x,y\in B,\, x\neq y}\frac{|\partial_{i_1\ldots i_s}u(x)-\partial_{i_1\ldots i_s}u(y)|}{|x-y|^\lambda}\\ &\leq K\sum_{i_1,\ldots,i_{s+1}=1}^{m}\sup_{x,y\in B,\, x\neq y}\frac{|\partial_{i_1\ldots i_{s+1}}u(x)-\partial_{i_1\ldots i_{s+1}}u(y)|}{|x-y|^\lambda}\end{aligned}$$

Which proves $\|u\|_s^B \leq K\|u\|_{s+1}^B$.

Definition 3.11. *For a function $v=(v_1,\ldots,v_q):B\to\mathbb{R}^q$, we define its $C^{s,\lambda}$ norm to be*

$$\|v\|_s^B := \sum_{j=1}^{q}\|v_j\|_s^B$$

An important property that follows directly from the definitions is that for $0\leq r\leq s$,

$$\|u\|_s^B \leq \|u\|_0^B + \sum_{i_1,\ldots,i_r=1}\|\partial_{i_1\ldots i_r}u\|_{s-r}^B \leq \|u\|_r^B + \|u\|_s^B$$

Now, we extend the concept of Hölder norms to manifolds using *partitions of unity*:

Definition 3.12. *Given a manifold M with an open cover $\{U_\alpha\}$, a partition of unity subordinate to $\{U_\alpha\}$ is a collection of maps $\{\psi_\alpha : U_\alpha\to\mathbb{R}\}$ such that $\psi_\alpha\geq 0$, $\{\operatorname{supp}\psi_\alpha\}$ is locally finite, and $\sum_\alpha\psi_\alpha=1$.*

Definition 3.13. *Given a manifold M with a $\{\psi_\alpha\}$ subordinate to a C^∞ atlas $\{U_\alpha\}$:*

$$\|v\|_s = \sum_{\alpha,j}\|\psi_\alpha v_j\|_s^B$$

Property (7) holds true on manifolds as well:

$$\|u\|_s \leq K\|u\|_{s+1}^B \tag{8}$$

Our norms have two other useful properties: For two functions u, v on M, and for $0 \leq r < s$,

$$\|uv\|_s \leq C_s \|u\|_s \|v\|_s \tag{9}$$

$$\|uv\|_s \leq K(\|u\|_s\|v\|_r + \|u\|_r\|v\|_s) + C_s\|u\|_{s-1}\|v\|_{s-1} \tag{10}$$

Now that we have defined the proper norms and listed a few useful properties, we are ready to tackle the geometric PDE problem of isometric embeddings.

4. **Günther's Solution**

We have now arrived at the center of our discussion, Günther's solution to Nash's isometric Embedding Problem. Günther's perturbation result comes in two versions, Theorem 4.1 and Theorem 4.12. We will discuss in detail the proof of Theorem 4.1, then we will briefly sketch the proof of variant Theorem 4.12 and indicate how it can be applied to the isometric embedding theorem, which requires Theorem 4.15.

4.1. *Günther's First Perturbation Result and the Associated Fixed-Point Problem.* From now until the end of Section 4.4, we assume that M is a *compact* Riemannian manifold. We define $E(u)(h, f)$ for a free mapping $u \in C^\infty(M, \mathbb{R}^N)$, $h \in C^{s,\lambda}(M, T^*M)$, and $f \in C^{s,\lambda}(M, \mathrm{Sym}^2(T^*M))$ to be the unique solution v to the system

$$\nabla_i u \cdot v = h_i \quad \nabla_i \nabla_j u \cdot v = f_{ij} \tag{11}$$

so that for all $x \in M$, $\|v(x)\|$ is minimal. One can define $v = E(u)(h, f)$ point-wise by pseudo-inverse.

Given our linear system, we can write $v = (v_1, \ldots, v_N)$ as

$$v_\alpha = A^{\alpha,i}(u)h_i + B^{\alpha,ij}(u)f_{ij} \quad \alpha = 1, \ldots, N$$

where $A^{\alpha,i}(u)$ is a smooth contravariant vector field and $B^{\alpha,ij}(u)$ is a smooth symmetric $(0, 2)$ tensor field on M. Then, define

$$D(u) := \max\left(\sum_{\alpha=1}^{N} \|A^\alpha(u)\|_2, \sum_{\alpha=1}^{N} \|B^\alpha(u)\|_2\right)$$

Günther's first perturbation result states:

Theorem 4.1 ([28] Theorem / [27] Theorem 1). *Let M be a compact Riemannian manifold. For a free mapping $u \in C^\infty(M, \mathbb{R}^N)$, $f \in C^{s,\lambda}(M, \mathrm{Sym}^2(T^*M))$ with $s \geq 2$ (resp. $f \in C^\infty(M, \mathrm{Sym}^2(T^*M))$), there exists some $\theta > 0$ independent of u, s, f such that if*

$$D(u)\|E(u)(0, f)\|_2 \leq \theta$$

then there exists a $v \in C^{s,\lambda}(M, \mathbb{R}^N)$ *(resp.* $v \in C^\infty(M, \mathbb{R}^N)$*) with* $\|v\|_2 \le \|E(u)(0, f)\|_2$ *such that*

$$\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v + \partial_i v \cdot \partial_j v = f_{ij}$$

The purpose of the next few sections is to build up to the proof of this theorem, which will be provided in Section 4.4.

To prove this theorem, Günther's strategy was to express v as a solution to a fixed-point problem, as motivated by the following definition and theorem[22]:

Definition 4.2. *Let* (X, d) *be a metric space. Then* $\mathcal{F} : X \to X$ *is a contraction mapping if there exists a Lipschitz constant* $\theta < 1$ *such that* $d(\mathcal{F}(x), \mathcal{F}(y)) \le \theta d(x, y)$ *for all* $x, y \in X$*.*

Theorem 4.3 (Fixed Point Theorem, Theorem 5.1 in [23]). *If* (X, d) *is a nonempty complete metric space and* $E : X \to X$ *is a contraction mapping, then* E *admits a unique fixed point* y *such that* $E(y) = y$*, which may be realized as the limit of the recursively defined sequence* $x_{n+1} = E(x_n)$*.*

Günther's "contraction mapping" is the pseudoinverse $E(u)$: By Lemma 4.10, which will be proven in Section 4.3, we are able to express the desired variation v as a solution to a fixed-point problem $v = E(u)(h(v), f(v))$. Proving Theorem 4.1 essentially amounts to finding a Lipschitz-type constant θ.

We will assume the Hölder condition–that is, wherever λ appears, $0 < \lambda < 1$. We need the completeness of Hölder spaces to prove the convergence of a recursive sequence of the form $v_{k+1} = E(u)(h(v_k), f(v_k))$ by applying the Arzela–Ascoli theorem.

Critical in this set up is that the LHS of (11) are *linear* in v; recall that in the previous section, the system (6) was linear not in v but in the infinitesimal variation w. Because u is a free mapping, all the derivatives $\nabla_i u$ and $\nabla_i \nabla_j u$ are linearly independent, so solutions exist if the ambient dimension N is large enough, given that the RHS is in the correct subspace. Günther realized that if one can express our desired variation v as a solution to such a fixed-point problem, one avoids *any* loss in differentiability.

Once we know that we can express the variation v in this way, we need to be able to estimate v in the correct differentiability class. In other words, we

[22] In [23], the statement of Theorem 4.3 is given for Banach spaces, but observing the proof reveals that the structure of a normed vector space is not required; the proof works for any nonempty complete metric space.

need to be able to bound the pseudo-inverse. We observe that

$$\|v_\alpha\|_2 \leq \|A^{\alpha,i}(u)h_i\|_2+\|B^{\alpha,ij}(u)f_{ij}\|_2 \leq C_2\|A^{\alpha,i}(u)\|_2\|h_i\|_2+C_2\|B^{\alpha,ij}(u)\|_2\|f_{ij}\|_2$$

$$\implies \|E(u)(h,f)\|_2 \leq KD(u)(\|h\|_2 + \|f\|_2) \tag{12}$$

This inequality almost generalizes for $s > 2$:

Lemma 4.4 ([28] Lemma 5). *For our current assumptions on u, f, and h and $s \geq 3$,*

$$\|E(u)(h,f)\|_s \leq KD(u)(\|h\|_s + \|f\|_s) + C(u,s)(\|h\|_{s-1} + \|f\|_{s-1})$$

K is independent of u and s.

Proof. Note that by (10) and (8),

$$\begin{aligned}\|A^\alpha(u)h\|_s &\leq K\left(\|A^\alpha(u)\|_s \|h\|_2 + \|A^\alpha(u)\|_2 \|h\|_s\right) + C_s \|A^\alpha(u)\|_{s-1} \|h\|_{s-1} \\ &\leq KD(u)\|h\|_s + C(u,s)\|h\|_{s-1}\end{aligned}$$

Analogously, $\|B^\alpha(u)f\|_s \leq KD(u)\|f\|_s + C(u,s)\|f\|_{s-1}$. So we can conclude that

$$\|E(u)(h,f)\|_s \leq \|A^\alpha(u)h\|_s+\|B^\alpha(u)f\|_s \leq KD(u)(\|h\|_s+\|f\|_s)+C(u,s)(\|h\|_{s-1}+\|f\|_{s-1})$$

□

4.2. *Strictly Elliptic Operators and The Screened Poisson Equation.* Elliptic operators, which are particular types of 2nd order differential operators, are nice because much is known about their solutions to their associated PDE's. The convenient properties of the Laplacian as an elliptic operator are key to Günther's proof of Theorem 4.1.

$$\Delta x - \nu x = f \quad \nu \geq 0 \tag{13}$$

Consider equation (13) on the unit ball in Euclidean space. This is known as the *screened Poisson equation.* Given boundary conditions, the above PDE has a unique solution, so the operator $\Delta - \nu\mathbb{1}$ has a well-defined bounded inverse $(\Delta - \nu\mathbb{1})^{-1} : C^{s,\lambda} \to C^{s+2,\lambda}$.

This is true in general for a class of *strictly elliptic operators* $\mathcal{L}$, defined below:

Definition 4.5. *A second-order differential operator*

$$\mathcal{L}u = a_{ij}\partial_{ij}u + b_i\partial_i u + cu$$

is strictly elliptic if a_{ij} satisfy $a_{ij}(x)\xi_i\xi_j \geq \kappa\|\boldsymbol{\xi}\|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^n$ and some $\kappa > 0$.

We have the following result for strictly elliptic operators:

THEOREM 4.6. *If $\mathcal{L}$ is a strictly elliptic 2nd order differential operator*

$$\mathcal{L}u := a_{ij}\partial_{ij}u + b_i\partial_i u + cu$$

satisfying $\|a_{ij}\|_0, \|b_i\|_0, \|c\|_0 \le \Lambda$ for some $\Lambda > 0$, then for a bounded open set $E \subset \mathbb{R}^n$ with smooth boundary, then there exists $\eta \ge 0$ such that for $\nu \ge \eta$, the associated Dirichlet problem

$$\begin{cases} \mathcal{L}u - \nu u = f & \bar{E} \\ u = \varphi & \partial E \end{cases}$$

has a unique solution $u \in C^{s+2,\lambda}$ for $f \in C^{s,\lambda}$.

To understand why this is true, see for instance Theorem 3 of Section 6.2.2 in [21], which shows uniqueness, and Schauder's global estimate, which shows that the solution is in the right differentiability class (Theorem 6.6 in [23]). Necessary for Schauder's global estimate is the interior estimate, which we will state and use shortly.

On our compact Riemannian manifold M, we let g_0 be an auxiliary metric on M and define the covariant Laplacian on tensors by[23] $\Delta = \nabla^k\nabla_k = g_0^{kl}\nabla_l\nabla_k$. As in the case of $\mathbb{R}^n$, $\Delta - \mathbb{1}$ on M also has a well-defined bounded inverse $G := (\Delta - \mathbb{1})^{-1}$.[24] Recall that by definition of bounded operators on Banach spaces, we have for any $C^{s,\lambda}$ tensor field y,

$$\|Gy\|_{s+2} \le C_s\|y\|_s \tag{14}$$

We can get more specific estimates on G by applying an important and useful result about elliptic operators:

THEOREM 4.7 (Schauder's Interior Estimate, Theorem 6.2 in [23]). *If $\mathcal{L}$ is a strictly elliptic 2nd order differential operator on Euclidean functions of n variables*

$$\mathcal{L}u := a_{ij}\partial_{ij}u + b_i\partial_i u + cu$$

satisfying $\|a_{ij}\|_0, \|b_i\|_0, \|c\|_0 \le \Lambda$ for some $\Lambda > 0$, then for a bounded open set $E \subset \mathbb{R}^n$, if $u \in C^2(E)$ solves

$$\mathcal{L}u = f$$

for some $f \in C^{0,\lambda}$, then $u \in C^{2,\lambda}$ and

$$\|u\|_2^E \le K(n, \lambda, \kappa, \Lambda)\left(\|u\|_0^E + \|f\|_0^E\right)$$

[23] We use the Einstein summation convention here and in the proofs of this section.

[24] In the original paper, Günther used the Laplace operator defined by Lichnerowicz in [35]; we will be using the covariant Laplacian, as used in [2].

We can apply Schauder's interior estimate to manifolds by identifying coordinate neighborhoods on M with open sets of $\mathbb{R}^n$. We get the following result:

LEMMA 4.8 ([28] Lemma 3). *For* $h \in C^{s,\lambda}(M, T^*M)$ *and* $f \in C^{s,\lambda}(\mathrm{Sym}^2(T^*M))$ *with* $s \geq 1$,

$$\|Gh\|_{s+2} \leq K\|h\|_s + C_s\|h\|_{s-1}$$

$$\|Gf\|_{s+2} \leq K\|f\|_s + C_s\|f\|_{s-1}$$

Proof. I prove this for $f \in \mathrm{Sym}^2(T^*M))$ using local coordinates; the proof for vector fields is analogous.

As before, let $\{(U_\alpha, \varphi_\alpha)\}$ be an atlas, and $\{\psi_\alpha\}$ be a smooth partition of unity subordinate to $\{U_\alpha\}$. The covariant Laplace operator is written in local coordinates as $g_0^{ab}\partial_a\partial_b$. Let $z = Gf$ and $\partial^{(s)} = \partial_{i_1}\ldots\partial_{i_s}$ for an arbitrary collection of indices $i_1, \ldots, i_s$.

$$\begin{aligned}
&g_0^{ab}\partial_a\partial_b\partial^{(s)}(\psi_\alpha z_{ij}) \\
&= \partial_{i_s}\left(g_0^{ab}\partial^{(s-1)}\partial_a\partial_b(\psi_\alpha z_{ij})\right) - \left(\partial_{i_s}g_0^{ab}\right)\partial^{(s-1)}\partial_a\partial_b(\psi_\alpha z_{ij}) \\
&= \partial_{i_s}\left(\partial_{i_{s-1}}\left(g_0^{ab}\partial^{(s-2)}\partial_a\partial_b(\psi_\alpha z_{ij})\right)\right) - \left(\partial_{i_s}\partial_{i_{s-1}}g_0^{ab}\right)\left(\partial^{(s-2)}\partial_a\partial_b(\psi_\alpha z_{ij})\right) \\
&\quad - \left(\partial_{i_{s-1}}g_0^{ab}\right)\left(\partial_{i_s}\partial^{(s-2)}\partial_a\partial_b(\psi_\alpha z_{ij})\right) - (\partial_{i_s}g_0^{ab})\partial^{(s-1)}\partial_a\partial_b(\psi_\alpha z_{ij}) \\
&\quad \vdots
\end{aligned}$$

In the expression after the first equals sign, the derivative is of order $s+1$ on z_{ij}, and the last three terms in the expression after the second equals sign have derivatives of order $\leq s+1$ on z_{ij}. This pattern continues as we move g_0^{ab} inside the derivatives using the product rule. This means

$$g_0^{ab}\partial_a\partial_b\partial^{(s)}(\psi_\alpha z_{ij}) = \partial^{(s)}(\psi_\alpha f_{ij}) + T_{ij}^{\alpha,(s)}z$$

where T_{ij}^{α} is a differential operator of $\leq s+1$.

Recall that we define the $C^{s,\lambda}$ norms on a (compact) manifold by $\|f\|_s := \sum_{\alpha,i,j}\|\psi_\alpha f_{ij}\|_s^B$. The norms defined under different partitions of unity are equivalent. Choosing a partition of unity $\tilde{\psi}_\alpha$ so that $\tilde{\psi}_\alpha(B') = 1$ for an open ball $B' \subseteq B$, we get $\|f\|_s^{B'} \leq \|\tilde{\psi}_\alpha f_{ij}\|_s^B \leq \|f\|_s$. Let B_1, B_2 be open balls so that $\operatorname{supp}\psi_\alpha \subset \bar{B}_1 \subseteq B_2 \subseteq \bar{B}_2 \subseteq B$. Similarly, we get

$$\sum_{(s)}\left\|T_{ij}^{A,(s)}z\right\|_0^{B_2} \leq \sum_{(s)}\left\|T_{ij}^{A,(s)}z\right\|_0 \leq C_s\|z\|_{s+1} \tag{15}$$

The Laplacian $g_0^{ab}\partial_a\partial_b$ is a linear elliptic differential operator, which means that we can apply Schauder's interior estimate (Theorem 4.7):

$$\|\partial^{(s)}(\psi_\alpha z_{ij})\|_2^{B_1} \le K\left(\left\|\partial^{(s)}(\psi_\alpha f_{ij}) + T_{ij}^{\alpha,(s)} z\right\|_0^{B_2} + \left\|\partial^{(s)}(\psi_\alpha z_{ij})\right\|_0^{B_2}\right) \tag{16}$$

Note that because g_0 is an auxiliary metric defined independently of u, the constant K also does not depend on u.

Applying definitions of norms, we get

$$\|\psi_\alpha z_{ij}\|_0^{B_1} \le \|z\|_{s+1} \tag{17}$$

$$\sum_{(s)} \|\partial^{(s)}(\psi_\alpha z_{ij})\|_0^{B_2} \le \|\psi_\alpha z_{ij}\|_s^B \le C_s\|z\|_{s+1} \tag{18}$$

By (15), (16), (17), and (18):

$$\begin{aligned}
&\|\psi_\alpha z_{ij}\|_{s+2}^B \\
&\le \|\psi_\alpha z_{ij}\|_0^{B_1} + \sum_{(s)} \|\partial^{(s)}(\psi_\alpha z_{ij})\|_2^{B_1} \\
&\le \|\psi_\alpha z_{ij}\|_0^{B_1} + K\sum_{(s)}\left(\left\|\partial^{(s)}(\psi_\alpha f_{ij}) + T_{ij}^{\alpha,(s)} z\right\|_0^{B_2} + \left\|\partial^{(s)}(\psi_\alpha z_{ij})\right\|_0^{B_2}\right) \\
&\le K\left(\|\psi_\alpha f_{ij}\|_0^{B_2} + \sum_{(s)}\left\|\partial^{(s)}(\psi_\alpha f_{ij})\right\|_0^{B_2}\right) + \|\psi_\alpha z_{ij}\|_0^{B_1} + K\sum_{(s)}\left(\left\|T_{ij}^{\alpha,(s)} z\right\|_0^{B_2} + \left\|\partial^{(s)}(\psi_\alpha z_{ij})\right\|_0^{B_1}\right) \\
&\le K\|\psi_\alpha f_{ij}\|_s^B + C_s\|z\|_{s+1}
\end{aligned}$$

$$\begin{aligned}
\implies \|Gf\|_{s+2} = \sum_{\alpha,i,j}\|\psi_\alpha z_{ij}\|_{s+2}^B &\le K\sum_{\alpha,i,j}\|\psi_\alpha f_{ij}\|_s + C_s\|Gf\|_{s+1} \\
&= K\|f\|_s + C_s\|Gf\|_{s+1} \le K\|f\|_s + C_s'\|f\|_{s-1}
\end{aligned}$$

□

4.3. *Günther's Lemmas.* Now, we provide the proofs of the lemmas necessary to set up the fixed-point problem for Theorem 4.1. The expression of the desired variation v as a solution to a fixed-point problem follows from Lemmas 4.9 and 4.10; Lemma 4.11 provides the remaining estimates necessary to solve the fixed-point problem.

First, we prove a useful identity. Now, define the vector field $N(v)$ and the symmetric $(0, 2)$ tensor field $L(v)$ by:

$$N_i(v) := -\Delta v \cdot \nabla_i v \tag{19}$$

(20)
$$L_{ij}(v) := 2\nabla^k\nabla_i v\cdot\nabla_k\nabla_j v - 2\Delta v\cdot\nabla_i\nabla_j v - \nabla_i v\cdot\nabla_j v + R_i{}^p\nabla_p v\cdot\nabla_j v + R_j{}^p\nabla_p v\cdot\nabla_i v$$

$$1 \le i, j \le n$$

The above two definitions are rather unintuitive, but as we will see in our first two lemmas, they are helpful in writing the desired variation as a solution to a fixed-point problem.

LEMMA 4.9 ([28] Lemma 1). *For* $u \in C^\infty(M, \mathbb{R}^N)$, $v \in C^2(M, \mathbb{R}^N)$, *and* $1 \le i, j \le n$,

$$\begin{aligned}
&(\Delta - \mathbb{1})(\nabla_i u\cdot\nabla_j v + \nabla_j u\cdot\nabla_i v + \nabla_i v\cdot\nabla_j v)\\
=&\ \nabla_i\left((\Delta-\mathbb{1})(\nabla_j u\cdot v) - N_j(v)\right) + \nabla_j\left((\Delta-\mathbb{1})(\nabla_i u\cdot v) - N_i(v)\right)\\
&- 2(\Delta-\mathbb{1})(\nabla_i\nabla_j u\cdot v) + L_{ij}(v) + R_j{}^p\nabla_p(\nabla_i u\cdot v) + R_i{}^p\nabla_p(\nabla_j u\cdot v)
\end{aligned}$$

Remark. Because the Laplacian does not commute with covariant differentiation, we get some curvature terms:[25]

$$\begin{aligned}
&[\Delta, \nabla_j]\\
=&g_0^{kl}\nabla_k\nabla_l\nabla_j - \nabla_j g_0^{kl}\nabla_k\nabla_l\\
=&g_0^{kl}\left(\nabla_k\nabla_l\nabla_j - \nabla_j\nabla_k\nabla_l\right)\\
=&g_0^{kl}\left(\nabla_k[\nabla_l,\nabla_j] + \nabla_k\nabla_j\nabla_l - \nabla_j\nabla_k\nabla_l\right)\\
=&g_0^{kl}\left(\nabla_k[\nabla_l,\nabla_j] + [\nabla_k,\nabla_j]\nabla_l\right)
\end{aligned}$$

For a sufficiently differentiable function a, we get

$$\begin{aligned}
&\implies [\Delta,\nabla_j]a\\
=&g_0^{kl}\left(\nabla_k[\nabla_l,\nabla_j]a + [\nabla_k,\nabla_j]\nabla_l a\right)\\
=&g_0^{kl}\left(R_{kjl}{}^p\nabla_p a\right)\\
=&g_0^{kl}\left(R_{kjlq}g_0^{pq}\nabla_p a\right)\\
=&R_{jq}g_0^{pq}\nabla_p a\\
=&R_j{}^p\nabla_p a
\end{aligned}$$

[25]That the metric commutes with the covariant derivative is equivalent to the metric compatibility condition on the Levi–Civita connection:

$$\nabla_Z(g_0(X,Y)) = Z(g_0(X,Y)) - g_0(\nabla_Z X, Y) - g_0(X, \nabla_Z Y) = 0$$

So in particular, when expanding $\nabla_j(g_0^{kl}\nabla_k\nabla_l) = (\nabla_j g_0^{kl})\nabla_k\nabla_l + g_0^{kl}\nabla_j\nabla_k\nabla_l$, the term $\nabla_j g_0^{kl}\nabla_k\nabla_l$ vanishes because $\nabla_j g_0^{kl} = 0$.

Regarding the proof, some relevant calculations are shown in Section 6.3 of [2]. Nevertheless, we go step-by-step in the proof below.

Proof. Using the product rule, we have:

$$\nabla_j(\nabla_i u \cdot v) + \nabla_i(\nabla_j u \cdot v) - 2\nabla_i\nabla_j u \cdot v = \nabla_i u \cdot \nabla_j v + \nabla_j u \cdot \nabla_i v$$

Applying this observation as well as the commutation formula from the remark, we get

$$\begin{aligned}&(\Delta - \mathbb{1})(\nabla_j(\nabla_i u \cdot v) + \nabla_i(\nabla_j u \cdot v))\\ &= \nabla_j((\Delta - \mathbb{1})(\nabla_i u \cdot v)) + R_j{}^p\nabla_p(\nabla_i u \cdot v) + \nabla_i((\Delta - \mathbb{1})(\nabla_j u \cdot v)) + R_i{}^p\nabla_p(\nabla_j u \cdot v)\end{aligned}$$

$$\begin{aligned}&(\Delta - \mathbb{1})(\nabla_i v \cdot \nabla_j v)\\ &= \nabla^k(\nabla_k\nabla_i v \cdot \nabla_j v + \nabla_i v \cdot \nabla_k \nabla_j v) - \nabla_i v \cdot \nabla_j v\\ &= \Delta\nabla_i v \cdot \nabla_j v + 2\nabla_k\nabla_i v \cdot \nabla^k\nabla_j v + \nabla_i v \cdot \Delta\nabla_j v - \nabla_i v \cdot \nabla_j v\\ &= \nabla_i(\Delta v \cdot \nabla_j v) + \nabla_j(\Delta v \cdot \nabla_i v) - 2\Delta v \cdot \nabla_i\nabla_j v + R_i{}^p\nabla_p v \cdot \nabla_j v + R_j{}^p\nabla_p v \cdot \nabla_i v\\ &\quad + 2\nabla_k\nabla_i v \cdot \nabla^k\nabla_j v - \nabla_i v \cdot \nabla_j v\end{aligned}$$

Recalling the definition of $N(v)$ and $L(v)$ and combining the above calculations we get

$$\begin{aligned}&(\Delta - \mathbb{1})(\nabla_i u \cdot \nabla_j v + \nabla_j u \cdot \nabla_i v + \nabla_i v \cdot \nabla_j v)\\ &= (\Delta - \mathbb{1})(\nabla_j(\nabla_i u \cdot v) + \nabla_i(\nabla_j u \cdot v) - 2\nabla_i\nabla_j u \cdot v + \nabla_i v \cdot \nabla_j v)\\ &= \nabla_i\left((\Delta - \mathbb{1})(\nabla_j u \cdot v) - N_j(v)\right) + \nabla_j\left((\Delta - \mathbb{1})(\nabla_i u \cdot v) - N_i(v)\right)\\ &\quad - 2(\Delta - \mathbb{1})(\nabla_i\nabla_j u \cdot v) + L_{ij}(v) + R_j{}^p\nabla_p(\nabla_i u \cdot v) + R_i{}^p\nabla_p(\nabla_j u \cdot v)\end{aligned}$$

□

Now, define the symmetric $(0,2)$-tensor field M:

$$M_{ij}(v) = L_{ij}(v) + R_j{}^p\nabla_p\left(GN(v)\right)_i + R_i{}^p\nabla_p\left(GN(v)\right)_j \tag{21}$$

Lemma 4.10 ([28] Lemma 2). *For $u \in C^\infty(M, \mathbb{R}^N)$, $v \in C^{2,\lambda}(M, \mathbb{R}^N)$, $f \in C^{2,\lambda}(\mathrm{Sym}^2(T^*M))$, and $1 \le i, j \le n$, if*

$$\nabla_i u \cdot v = (GN(v))_i \qquad \nabla_i\nabla_j u \cdot v = -\frac{1}{2}f_{ij} + \frac{1}{2}(GM(v))_{ij}$$

Then we have

$$\nabla_i u \cdot \nabla_j v + \nabla_j u \cdot \nabla_i v + \nabla_i v \cdot \nabla_j v = f_{ij}$$

Proof. If we make these substitutions in the identity from Lemma 4.9,

$$\nabla_i\left((\Delta-\mathbb{1})(\nabla_j u\cdot v)-N_j(v)\right)+\nabla_j\left((\Delta-\mathbb{1})(\nabla_i u\cdot v)-N_i(v)\right)$$

vanishes. Then we have

$$\begin{aligned}
&(\Delta-\mathbb{1})\left(\nabla_i u\cdot\nabla_j v+\nabla_j u\cdot\nabla_i v+\nabla_i v\cdot\nabla_j v\right)\\
=&\ -2(\Delta-\mathbb{1})(\nabla_i\nabla_j u\cdot v)+M_{ij}(v)\\
=&\,(\Delta-\mathbb{1})\left(f_{ij}-(GM(v))_{ij}\right)+M_{ij}(v)\\
=&\,(\Delta-\mathbb{1})(f_{ij})\\
&\implies \nabla_i u\cdot\nabla_j v+\nabla_j u\cdot\nabla_i v+\nabla_i v\cdot\nabla_j v=f_{ij}
\end{aligned}$$

□

By this lemma, we can prove Theorem 4.1 by solving the following fixed-point problem:

$$v=E(u)\left(GN(v),\ -\frac{1}{2}f+\frac{1}{2}GM(v)\right)$$

Now, we provide estimates for N and M:

LEMMA 4.11 ([28] Lemma 4). *For $v\in C^{s+2,\lambda}(M,\mathbb{R}^N)$ with $s\geq 0$,*

$$\|N(v)\|_s,\ \|M(v)\|_s\leq K\|v\|_2\|v\|_{s+2}+C_s\|v\|_{s+1}^2$$

For $v_1,v_2\in C^{2,\lambda}(M,\mathbb{R}^N)$,

$$\|N(v_1)-N(v_2)\|_0,\ \|M(v_1)-M(v_2)\|_0\leq K(\|v_1\|_2+\|v_2\|_2)\|v_1-v_2\|_2$$

Proof. Let $v\in C^{s+2,\lambda}(M,\mathbb{R}^N)$. Applying the definitions and properties of $C^{s,\lambda}$ norms gives us

$$\|\nabla_i\nabla_j v\|_0\leq K\|v\|_2 \tag{22}$$

$$\|\nabla_i\nabla_j v\|_s\leq K\|v\|_{s+2}+C_s\|v\|_{s+1} \tag{23}$$

We observe that by (23), (22), (10), and (8)

$$\begin{aligned}
&\|N(v)\|_s\\
=&\,\|\Delta v\cdot\nabla_i v\|_s\\
\leq&\, K(\|\Delta v\|_s\|\|\nabla_i v\|_0+\|\Delta v\|_0\|\nabla_i v\|_s)+C_s\|\Delta v\|_{s-1}\|\nabla_i v\|_{s-1}\\
\leq&\, K((K\|v\|_{s+2}+C_s\|v\|_{s+1})K\|v\|_1+K\|v\|_2\|v\|_{s+1})+C_s(K\|v\|_{s+1}+C_{s-1}\|v\|_s)K\|v\|_s\\
\leq&\, K'\|v\|_2\|v\|_{s+2}+C_s\|v\|_{s+1}^2
\end{aligned}$$

Next, we observe that by (22), (23), and (9),

$$\|\nabla_i v\cdot\nabla_j v\|_s\leq C_s\|v\|_{s+1}^2 \tag{24}$$

$$\|\nabla_p GN(v)\|_s \le K\|GN(v)\|_{s+1} \le K'\|N(v)\|_{s-1} \le K''\|v\|_{s+1}\|v\|_2 + C_s\|v\|_s^2$$

$$\Longrightarrow \|\nabla_p GN(v)\|_s \le K\|v\|_{s+1}\|v\|_2 + C_s\|v\|_s^2 \tag{25}$$

$$\begin{aligned}
&\|\nabla_i\nabla_j v \cdot \nabla_k\nabla_l v\|_s \\
&\le K(\|\nabla_i\nabla_j v\|_s\|\nabla_k\nabla_l v\|_0 + \|\nabla_i\nabla_j v\|_0\|\nabla_k\nabla_l v\|_s) + C_s\|\nabla_i\nabla_j v\|_{s-1}\|\nabla_i\nabla_j v\|_{s-1} \\
&\le K'\|v\|_{s+2}\|v\|_2 + C_s\|v\|_{s+1}^2
\end{aligned}$$

$$\Longrightarrow \|\nabla_i\nabla_j v \cdot \nabla_k\nabla_l v\|_s \le K\|v\|_{s+2}\|v\|_2 + C_s\|v\|_{s+1}^2 \tag{26}$$

These above are all of the types of terms that appear in the definition of $M(v)$. Some are multiplied by a Ricci curvature term, but because we are in the setting of compact manifolds, these terms are bounded, and the Ricci curvature is defined by the auxiliary metric g_0, independent of u. Hence we have by (24), (25), (26), and the triangle inequality that

$$\|M(v)\|_s \le K\|v\|_2\|v\|_{s+2} + C_s\|v\|_{s+1}^2$$

Now, we see that by triangle inequality, (9), and (22),

$$\begin{aligned}
&\|N(v_1) - N(v_2)\|_0 \\
&= \|\Delta v_1 \cdot \nabla_i v_1 - \Delta v_2 \cdot \nabla_i v_2\|_0 \\
&= \|\Delta(v_1 - v_2) \cdot \nabla_i v_1 + \Delta v_2 \cdot \nabla_i(v_1 - v_2)\|_0 \\
&\le \|\Delta(v_1 - v_2) \cdot \nabla_i v_1\|_0 + \|\Delta v_2 \cdot \nabla_i(v_1 - v_2)\|_0 \\
&\le K\|\Delta(v_1 - v_2)\|_0\|\nabla_i v_1\|_0 + K\|\Delta v_2\|_0\|\nabla_i(v_1 - v_2)\|_0 \\
&\le K\|v_1 - v_2\|_2(\|v_1\|_2 + \|v_2\|_2)
\end{aligned}$$

$$\Longrightarrow \|N(v_1) - N(v_2)\|_0 \le K\|v_1 - v_2\|_2(\|v_1\|_2 + \|v_2\|_2) \tag{27}$$

For very similar reasons

$$\begin{aligned}
&\|\nabla_i v_1 \cdot \nabla_j v_1 - \nabla_i v_2 \cdot \nabla_j v_2\|_0 \\
&= \|\nabla_i(v_1 - v_2) \cdot \nabla_j v_1 + \nabla_i v_2 \cdot \nabla_j(v_1 - v_2)\|_0 \\
&\le K\|v_1 - v_2\|_2(\|v_1\|_2 + \|v_2\|_2)
\end{aligned}$$

$$\begin{aligned}
&\|\nabla_i\nabla_j v_1 \cdot \nabla_k\nabla_l v_1 - \nabla_i\nabla_j v_2 \cdot \nabla_k\nabla_l v_2\|_0 \\
&= \|\nabla_i\nabla_j(v_1 - v_2) \cdot \nabla_k\nabla_l v_1 + \nabla_i\nabla_j v_2 \cdot \nabla_k\nabla_l(v_1 - v_2)\|_0 \\
&\le K\|v_1 - v_2\|_2(\|v_1\|_2 + \|v_2\|_2)
\end{aligned}$$

By (27) and (14), we see that

$$\begin{aligned}\|\nabla_p GN(v_1) - \nabla_p GN(v_2)\|_0 &\leq K\|GN(v_1) - GN(v_2)\|_1 \\ &\leq C_0\|N(v_1) - N(v_2)\|_0 \\ &\leq K\|v_1 - v_2\|_2(\|v_1\|_2 + \|v_2\|_2)\end{aligned}$$

Again, these are all of the types of terms that appear in the definition of $M(v)$, so we get

$$\|M(v_1) - M(v_2)\|_0 \leq K\|v_1 - v_2\|_2(\|v_1\|_2 + \|v_2\|_2)$$

□

4.4. *Proving the First Perturbation Result.* We have found all necessary estimations and are now ready to prove the first perturbation result.

Proof of Theorem 4.1. Recall that by Lemma 4.10, this theorem may be proved by solving the following fixed point problem:

$$v = E(u)\left(GN(v),\ -\frac{1}{2}f + \frac{1}{2}GM(v)\right)$$

We use a method of iteration. Define the following sequence:

$$v_0 = 0 \quad v_{k+1} = E(u)\left(GN(v_k),\ -\frac{1}{2}f + \frac{1}{2}GM(v_k)\right)$$

It suffices to prove that there exists a θ so that the sequence v_k converges in $C^{s,\lambda}$ for $s \geq 2$.

First, assume $s = 2$. $E(u)$ functions as a pseudo-inverse, so $E(u)$ is $\mathbb{R}$-linear in $C^{s,\lambda}(M, T^*M) \oplus C^{s,\lambda}(M, \mathrm{Sym}^2(T^*M))$. As a result, we have the following triangle inequality:

$$\|E(u)(h^1 + h^2,\ f^1 + f^2)\|_2 \leq \|E(u)(h^1, f^1)\|_2 + \|E(u)(h^2, f^2)\|_2 \tag{28}$$

Using (12), (28), Lemma 4.11, (14), and (8), one can obtain:

$$
\begin{aligned}
&\|v_{k+1}\|_2 \\
&= \left\| E(u)\left(GN(v_k),\ -\frac{1}{2}f + \frac{1}{2}GM(v_k)\right)\right\|_2 \\
&\le \left\| E(u)\left(GN(v_k),\ \frac{1}{2}GM(v_k)\right)\right\|_2 + \frac{1}{2}\,\|E(u)(0,f)\|_2 \\
&\le KD(u)\left(\|GN(v_k)\|_2 + \frac{1}{2}\,\|GM(v_k)\|_2\right) + \frac{1}{2}\|E(u)(0,f)\|_2 \\
&\le KD(u)\left(C_0\|N(v_k)\|_0 + C_0\|M(v_k)\|_0\right) + \frac{1}{2}\|E(u)(0,f)\|_2 \\
&\le KD(u)\left(2K\|v_k\|_2{}^2 + 2C_0'\|v_k\|_1{}^2\right) + \frac{1}{2}\|E(u)(0,f)\|_2 \\
&\le K_1D(u)\|v_k\|_2{}^2 + \frac{1}{2}\|E(u)(0,f)\|_2
\end{aligned}
$$

$$
\Longrightarrow \|v_{k+1}\|_2 \le K_1D(u)\|v_k\|_2{}^2 + \frac{1}{2}\|E(u)(0,f)\|_2 \tag{29}
$$

$\|E(u)(0,f)\|_{s'}$ is the determined by f, which is independent of the variation, so $\|E(u)(0,f)\|_{s'}$ is finite. Now, we prove by induction that if

$$
2K_1D(u)\|E(u)(0,f)\|_2 \le 1 \Longrightarrow \|v_k\|_2 \le \|E(u)(0,f)\|_2
$$

The base case $v_0 = 0$ is obvious; continuing inductively, we have by (29):

$$
\begin{aligned}
&\|v_{k+1}\|_2 \\
&\le K_1D(u)\|v_k\|_2{}^2 + \frac{1}{2}\|E(u)(0,f)\|_2 \\
&\le K_1D(u)\|E(u)(0,f)\|_2{}^2 + \frac{1}{2}\|E(u)(0,f)\|_2 \\
&\le K_1D(u)\cdot\frac{1}{2K_1D(u)}\|E(u)(0,f)\|_2 + \frac{1}{2}\|E(u)(0,f)\|_2 \\
&= \|E(u)(0,f)\|_2
\end{aligned}
$$

So for all k, we have

$$
\|v_k\|_2 \le \|E(u)(0,f)\|_2 \tag{30}
$$

Now, one can examine

$$
\|v_{k+1} - v_k\|_2 = \left\| E(u)\left(GN(v_k),\ -\frac{1}{2}f + \frac{1}{2}GM(v_k)\right) - E(u)\left(GN(v_{k-1}),\ -\frac{1}{2}f + \frac{1}{2}GM(v_{k-1})\right)\right\|_2
$$

Let $x(v_k) = GN(v_k)$ and $y(v_k) = -\frac{1}{2}f + \frac{1}{2}GM(v_k)$. We use the trick of adding and subtracting $E(u)(x(v_{k-1}), y(v_k))$:

$$\begin{aligned}&\|v_{k+1} - v_k\|_2 \\ &\le \|E(u)(x(v_k), y(v_k)) - E(u)(x(v_{k-1}), y(v_k))\|_2 + \|E(u)(x(v_{k-1}), y(v_k)) - E(u)(x(v_{k-1}), y(v_{k-1}))\|_2\end{aligned}$$

Then, I claim that

$$\|E(u)(h^1, f) - E(u)(h^2, f)\|_2 \le C_2 D(u)\|h^1 - h^2\|_2 \tag{31}$$

$$\|E(u)(h, f^1) - E(u)(h, f^2)\|_2 \le C_2 D(u)\|f^1 - f^2\|_2 \tag{32}$$

To see (31) and (32), recall that any solution $v = E(u)(h, f)$ can be written as

$$v_\alpha = A^{\alpha,i}(u)h_i + B^{\alpha,ij}(u)f_{ij}$$

Since $A^{\alpha,i}$ and $B^{\alpha,ij}$ depend only on u, by (9),

$$\begin{aligned}&\|E(u)(h^1, f) - E(u)(h^2, f)\|_2 \\ &= \|(A^\alpha(u)h^1 + B^\alpha(u)f) - (A^\alpha(u)h^2 + B^\alpha(u)f)\|_2 \\ &\le C_2\|A^\alpha(u)\|_2\|h^1 - h^2\|_2 \\ &\le C_2 D(u)\|h^1 - h^2\|_2\end{aligned}$$

$$\begin{aligned}&\|E(u)(h, f^1) - E(u)(h, f^2)\|_2 \\ &= \|(A^\alpha(u)h + B^\alpha(u)f^1) - (A^\alpha(u)h + B^\alpha(u)f^2)\|_2 \\ &\le C_2\|B^\alpha(u)\|_2\|f^1 - f^2\|_2 \\ &\le C_2 D(u)\|f^1 - f^2\|_2\end{aligned}$$

Now, it follows from (31), (32), (14), and Lemma 4.11 that

$$\begin{aligned}&\|v_{k+1} - v_k\|_2 \\ &\le \|E(u)(x(v_k), y(v_k)) - E(u)(x(v_{k-1}), y(v_k))\|_2 + \|E(u)(x(v_{k-1}), y(v_k)) - E(u)(x(v_{k-1}), y(v_{k-1}))\|_2 \\ &\le C_2 D(u)\,\|x(v_k) - x(v_{k-1})\|_2 + C_2 D(u)\,\|y(v_k) - y(v_{k-1})\|_2 \\ &= C_2 D(u)\,\|G(N(v_k) - N(v_{k-1}))\|_2 + \frac{1}{2}C_2 D(u)\,\|G(M(v_k) - M(v_{k-1}))\|_2 \\ &\le C_2 C_0^1 D(u)\|N(v_k) - N(v_{k-1})\|_0 + C_2 C_0^2 D(u)\|G(M(v_k) - M(v_{k-1}))\|_0 \\ &\le K_2 D(u)(\|v_k\|_2 + \|v_{k-1}\|_2)(\|v_k - v_{k-1}\|\|_2)\end{aligned}$$

Then, by (30), we can replace $\|v_k\|_2 + \|v_{k-1}\|_2$ with $2\|E(u)(0, f)\|_2$ to get:

$$\|v_{k+1} - v_k\|_2 \le 2K_2 D(u)\|E(u)(0, f)\|_2\,\|v_k - v_{k-1}\|_2$$

Then if $4K_2 D(u)\|E(u)(0, f)\|_2 \le 1$, then the sequence $\{v_k\}$ is Cauchy, and by completeness of the Hölder space $C^{2,\lambda}(M, \mathbb{R}^N)$, $\{v_k\}$ converges in $C^{2,\lambda}$ to some $v = E(u)\left(GN(v_k), -\frac{1}{2}f + \frac{1}{2}GM(v)\right)$.

To show that $v \in C^{s,\lambda}(M, \mathbb{R}^N)$, by Arzela–Ascoli, it suffices to show that the sequence $\{v_k\}_{k\geq 0}$ is bounded in $C^{s,\lambda}(M, \mathbb{R}^N)$, since we already have that $v_k \xrightarrow{C^{2,\lambda}} v$.

This may be demonstrated inductively. The base case $C^{2,\lambda}$ is already done. Now, suppose the sequence is bounded in $C^{k,\lambda}$ for $k \leq s' - 1$. Using (28), Lemma 4.4, Lemma 4.8, and (8), we get

$$\begin{aligned}
\|v_{k+1}\|_{s'} &= \left\| E(u)\left(GN(v_k),\ -\frac{1}{2}f + \frac{1}{2}GM(v_k)\right)\right\|_{s'} \\
&\leq \left\| E(u)\left(GN(v_k),\ \frac{1}{2}GM(v_k)\right)\right\|_{s'} + \frac{1}{2}\|E(u)(0,f)\|_{s'} \\
&\leq K_3' D(u)\left(\|GN(v_k)\|_{s'} + \frac{1}{2}\|GM(v_k)\|_{s'}\right) + C'(u,s)\left(\|GN(v_k)\|_{s'-1} + \frac{1}{2}\|GM(v_k)\|_{s'-1}\right) + \frac{1}{2}\|E(0,f)\|_{s'} \\
&\leq K_3'' D(u)\left(K\|N(v_k)\|_{s'-2} + C_s\|N(v_k)\|_{s'-3} + \frac{1}{2}K\|M(v_k)\|_{s'-2} + \frac{1}{2}C_s\|M(v_k)\|_{s'-3}\right) \\
&+ C'(u,s)\left(K\|N(v_k)\|_{s'-3} + C_s\|N(v_k)\|_{s'-4} + \frac{1}{2}K\|M(v_k)\|_{s'-3} + \frac{1}{2}C_s\|M(v_k)\|_{s'-4}\right) + \frac{1}{2}\|E(0,f)\|_{s'} \\
&\leq K_3'' D(u)\left(K'\|v_k\|_{s'}\|v_k\|_2 + C_2\|v_k\|_{s'-1}^2\right) + C'(u,s)\left(K'\|v_k\|_{s'-1}\|v\|_2 + C_2\|v_k\|_{s'-2}^2\right) + \frac{1}{2}\|E(0,f)\|_{s'} \\
&\leq K_3 D(u)(\|E(u)(0,f)\|_2\|v_k\|_{s'}) + C(u,s)(\|v_k\|_{s'-1}^2) + \frac{1}{2}\|E(u)(0,f)\|_{s'}
\end{aligned}$$

Now, if $2K_3D(u)\|E(u)(0,f)\|_2 \leq 1$, then we get by repeated substitution as below:

$$\begin{aligned}
&\|v_{k+1}\|_{s'} \\
&\leq \frac{1}{2}\|v_k\|_{s'} + C(u,s)\|v_k\|_{s'-1}^2 + \frac{1}{2}\|E(u)(0,f)\|_{s'} \\
&\leq \frac{1}{2}\left(\frac{1}{2}\|v_{k-1}\|_{s'} + C(u,s)\|v_{k-1}\|_{s'-1}^2 + \frac{1}{2}\|E(u)(0,f)\|_{s'}\right) + C(u,s)\|v_k\|_{s'-1}^2 + \frac{1}{2}\|E(u)(0,f)\|_{s'} \\
&\ \ \vdots \\
&\leq 2C(u,s)\sup_{l\geq 1}\|v_l\|_{s'-1}^2 + \|E(u)(0,f)\|_{s'}
\end{aligned}$$

By inductive hypothesis, $\sup_{l\geq 1}\|v_l\|_{s'-1}^2$ exists, and $\|E(u)(0,f)\|_{s'}$ is a constant. Therefore, the sequence $\{v_k\}$ is also bounded in $C^{s',\lambda}(M,\mathbb{R})$, which means by induction, $\{v_k\}$ is bounded in $C^{s,\lambda}(M,\mathbb{R})$ for any $s \geq 3$, given that $2K_3D(u)\|E(u)(0,f)\|_2 \leq 1$.

We have proven:

(i) If $2K_1D(u)\|E(u)(0,f)\|_2 \leq 1$ and $4K_2D(u)\|E(u)(0,f)\|_2 \leq 1$, the sequence $\{v_k\}$ converges in $C^{2,\lambda}(M,\mathbb{R}^N)$.

(ii) If $2K_3D(u)\|E(u)(0,f)\|_2 \leq 1$, the $\{v_k\}$ is bounded $C^{s,\lambda}$ for every $s \geq 3$.

Therefore, we set

$$\theta = \frac{1}{\max(2K_1, 4K_2, 2K_3)}$$

□

4.5. *How to Apply the Perturbation Result to the Original Isometric Embedding Problem.* During this time, we have been assuming that N is of "high enough dimension," but it is now natural to ask how high N needs to be.

Using his trick involving "Y and Z embeddings," Nash proved that the existence of a free embedding is guaranteed if $N \geq \frac{n(n+5)}{2}$ [42, pp. 52-59]. Alternatively, this is proven via a spherical fibre bundle argument by Mikhail Gromov and Vladimir Rokhlin in 2.5.3 of [26].[26] There are $\frac{n(n+3)}{2}$ first and second derivatives, but it turns out that assuming $N \geq \frac{n(n+3)}{2}$ is not enough to guarantee a solution to Günther's linearized problem. As indicated in the statement of Theorem 4.15, 5 extra dimensions are required.

Günther discovered that there is a variant of Theorem 4.1 applicable to the isometric embedding problem:

Theorem 4.12 ([27] Theorem 2). *Let $B \subseteq \mathbb{R}^n$ be the open unit ball and $B_1, B_2 \subseteq \mathbb{R}^n$ open sets with $\bar{B}_1 \subseteq B_2$, $\bar{B}_2 \subseteq B$. Let $u \in C^\infty(\bar{B}, \mathbb{R}^N)$ be a free mapping and $f = (f_{ij}) \in C^{s,\lambda}(B, \mathbb{R}^{n(n+1)/2})$ with $2 \leq s \leq \infty$ and* supp $f \subseteq B_1$.

There exists some $\theta > 0$ independent of u, s, f such that if

$$D(u)\|E(u)(0,f)\|_2 \leq \theta$$

then there exists a $v \in C^{s,\lambda}(B, \mathbb{R}^N)$ with supp $v \subseteq B_2$ *and*

$$\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v + \partial_i v \cdot \partial_j v = f_{ij}$$

To prove Theorem 4.12, we need to set up a fixed point problem analogous to that in the proof of Theorem 4.1, which means that we need new versions of Günther's lemmas from Section 4.3, all of which require our bounded operator $G : C^{s,\lambda} \to C^{s+2,\lambda}$, defined to be $(\Delta - \mathbb{1})^{-1}$. Fourtunately, because we are in the setting of the unit ball in Euclidean space, to find such an operator is

[26]This argument of Gromov-Rokhlin is explored more in detail of section 2.1 of Ben Andrews's notes [2].

easier; we can use simply the inverse of the Laplace operator Δ, which as we saw in Section 4.2, is well-defined and bounded.

Because of our current setting in the unit ball of $\mathbb{R}^n$, derivatives commute, so automatically by Schauder's interior estimate (Theorem 4.7), for every tensor y, we have an analogue of Lemma 4.8:

$$\|\Delta^{-1}y\|_{s+2} \leq K\|y\|_s$$

Günther employs the following trick in [27]: Let ρ be a smooth bump function such that $\operatorname{supp}\rho = B_2$ and $\rho(B_1) = \{1\}$. The condition that v takes the form $v = \rho^2 w$ for some bounded function w forces v to have compact support; recall that the manifolds for Theorem 4.1 are compact. Therefore, for the new theorem we are trying to uncover, v should have compact support as it did for Günther's lemmas and Theorem 4.1.

Analogously to (19), (20), (21), we define the vector field N and (0, 2) symmetric tensor field M:

$$N_j(w) := -\Delta^{-1}(\partial_j w \cdot \rho\Delta w) - (\partial_j\rho)(w \cdot w)$$

$$\begin{aligned}M_{ij}(w) :=& \frac{1}{2}\rho\Delta^{-1}((\Delta\rho)(\partial_i w \cdot \partial_j w) + 2\rho(\partial_{ki} w \cdot \partial_{kj} w)) + \rho\Delta^{-1}((\partial_k p)((\partial_{ki} w \cdot \partial_j w) + (\partial_i w \cdot \partial_{kj} w)))\\ &-\frac{1}{2}\rho\Delta^{-1}((\partial_i\rho)(\Delta w \cdot \partial_j w) + (\partial_j\rho)(\partial_i w \cdot \Delta w) + 2\rho(\Delta w \cdot \partial_{ij} w))\end{aligned}$$

The lemma below is a new version of Lemma 4.9:

Lemma 4.13. *Let $u \in C^\infty(\bar{B}, \mathbb{R}^N)$ and $v \in C^2(B, \mathbb{R}^N)$ of the form $v = \rho^2 w$ for some bounded function w and smooth bump function ρ with* $\operatorname{supp}\rho = B_2$ *and $\rho(B_1) = \{1\}$. For $1 \leq i, j \leq n$, on B_2, we have*

$$\begin{aligned}&\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v + \partial_i v \cdot \partial_j v\\ =&\rho^3\partial_j\left(\partial_i u \cdot \frac{w}{\rho} - N_i(w)\right) + \rho^3\partial_i\left(\partial_j u \cdot \frac{w}{\rho} - N_j(w)\right) - 2\rho^2\partial_{ij}u \cdot w + 2\rho^2 M_{ij}(w)\end{aligned}$$

Similarly, relevant calculations are shown in Section 6.5 of [2]; we provide them here as well.

Proof. Once again, using the product rule, we have

$$\partial_j(\partial_i u \cdot v) + \partial_i(\partial_j u \cdot v) - 2\partial_{ij}u \cdot v = \partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v$$

Then, we observe that

$$\begin{aligned}
&\partial_j(\partial_i u \cdot v) \\
=&\partial_{ji} u \cdot v + \partial_i u \cdot \partial_j v \\
=&\rho^3 \left(\partial_{ji} u \cdot \frac{w}{\rho}\right) + (3\rho\partial_j\rho)\,(\partial_i u \cdot w) + \rho^3 \left(\partial_i u \cdot \partial_j \left(\frac{w}{\rho}\right)\right) \\
=&\rho^3 \partial_j \left(\partial_i u \cdot \frac{w}{\rho}\right) + (3\rho\partial_j\rho)\,(\partial_i u \cdot w)
\end{aligned}$$

$$\begin{aligned}
&\partial_i v \cdot \partial_j v \\
=&(2\rho(\partial_i\rho)w + \rho^2\partial_i w) \cdot (2\rho(\partial_j\rho)w + \rho^2\partial_j w) \\
=&4\rho^2(\partial_i\rho)(\partial_j\rho)(w \cdot w) + 2\rho^3(\partial_i\rho)(w \cdot \partial_j w) + 2\rho^3(\partial_j p)(\partial_i w \cdot w) + \rho^4(\partial_i w \cdot \partial_j w)
\end{aligned}$$

We notice that

$$\begin{aligned}
&\Delta(\rho\partial_i w \cdot \partial_j w) \\
=&\partial_k(\partial_k\rho(\partial_i w \cdot \partial_j w)) + \partial_k(\rho(\partial_k(\partial_i w \cdot \partial_j w)) \\
=&(\Delta\rho)(\partial_i w \cdot \partial_j w)) + 2(\partial_k\rho)(\partial_k(\partial_i w \cdot \partial_j w)) + \rho(\Delta(\partial_i w \cdot \partial_j w)) \\
=&(\Delta\rho)(\partial_i w \cdot \partial_j w)) + 2(\partial_k\rho)(\partial_{ki} w \cdot \partial_j w) + 2(\partial_k\rho)(\partial_i w \cdot \partial_{kj} w) + \rho(\partial_k(\partial_{ki} w \cdot \partial_j w + \partial_i w \cdot \partial_{kj} w)) \\
=&(\Delta\rho)(\partial_i w \cdot \partial_j w)) + 2(\partial_k\rho)(\partial_{ki} w \cdot \partial_j w) + 2(\partial_k\rho)(\partial_i w \cdot \partial_{kj} w) + \rho(\partial_i\Delta w \cdot \partial_j w) \\
&+2\rho(\partial_{ki} w \cdot \partial_{kj} w) + \rho(\partial_i w \cdot \partial_j \Delta w) \\
=&(\Delta\rho)(\partial_i w \cdot \partial_j w)) + 2(\partial_k\rho)(\partial_{ki} w \cdot \partial_j w) + 2(\partial_k\rho)(\partial_i w \cdot \partial_{kj} w) + \partial_i(\rho\Delta w \cdot \partial_j w) \\
&+2\rho(\partial_{ki} w \cdot \partial_{kj} w) + \partial_j(\partial_i w \cdot \rho\Delta w) - (\partial_i\rho)(\Delta w \cdot \partial_j w) - (\partial_j\rho)(\partial_i w \cdot \Delta w) - 2\rho(\Delta w \cdot \partial_{ij} w)
\end{aligned}$$

So if we apply Δ^{-1} to the above expression and then multiply both sides by ρ^3, we have an alternative expression for $\rho^4(\partial_i w \cdot \partial_j w)$. We also observe the following product rule trick:

$$\frac{1}{2}\partial_i((\partial_j\rho)(w \cdot w)) - \frac{1}{2}(\partial_{ij}\rho)(w \cdot w) = (\partial_j\rho)(w \cdot \partial_i w)$$

So we get

$$
\begin{aligned}
&\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_j v + \partial_i v \cdot \partial_i v \\
=&\rho^3 \partial_j \left(\partial_i u \cdot \frac{w}{\rho} \right) + (3\rho \partial_j \rho)\,(\partial_i u \cdot w) + \rho^3 \partial_i \left(\partial_j u \cdot \frac{w}{\rho} \right) + (3\rho\partial_i\rho)\,(\partial_j u \cdot w) - 2\rho^2 \partial_{ij} u \cdot w \\
&+4\rho^2(\partial_i\rho)(\partial_j\rho)(w\cdot w) + 2\rho^3(\partial_i\rho)(w\cdot\partial_j w) + 2\rho^3(\partial_j p)(\partial_i w\cdot w) + \rho^3\partial_i(\Delta^{-1}(\rho\Delta w\cdot\partial_j w)) \\
&+\rho^3\partial_j(\Delta^{-1}(\partial_i w\cdot\rho\Delta w)) + \rho^3\Delta^{-1}((\Delta\rho)(\partial_i w\cdot\partial_j w) + 2\rho(\partial_{ki}w\cdot\partial_{kj}w)) \\
&+2\rho^3\Delta^{-1}((\partial_k p)((\partial_{ki}w\cdot\partial_j w) + (\partial_i w\cdot\partial_{kj}w))) - \rho^3\Delta^{-1}((\partial_i\rho)(\Delta w\cdot\partial_j w) + (\partial_j\rho)(\partial_i w\cdot\Delta w) + 2\rho(\Delta w\cdot\partial_{ij}w)) \\
=&\rho^3\partial_j\left(\partial_i u\cdot\frac{w}{\rho} + \Delta^{-1}(\partial_i w\cdot\rho\Delta w) + (\partial_i\rho)(w\cdot w)\right) + \rho^3\partial_i\left(\partial_j u\cdot\frac{w}{\rho} + \Delta^{-1}(\rho\Delta w\cdot\partial_j w) + (\partial_j\rho)(w\cdot w)\right) \\
&-2\rho^2\partial_{ij}u\cdot w + (3\rho\partial_j\rho)\,(\partial_i u\cdot w) + (3\rho\partial_i\rho)\,(\partial_j u\cdot w) + 4\rho^2(\partial_i\rho)(\partial_j\rho)(w\cdot w) - 2\rho^3(\partial_{ij}\rho)(w\cdot w) \\
&+\rho^3\Delta^{-1}((\Delta\rho)(\partial_i w\cdot\partial_j w) + 2\rho(\partial_{ki}w\cdot\partial_{kj}w)) + 2\rho^3\Delta^{-1}((\partial_k p)((\partial_{ki}w\cdot\partial_j w) + (\partial_i w\cdot\partial_{kj}w))) \\
&-\rho^3\Delta^{-1}((\partial_i\rho)(\Delta w\cdot\partial_j w) + (\partial_j\rho)(\partial_i w\cdot\Delta w) + 2\rho(\Delta w\cdot\partial_{ij}w)) \\
=&\rho^3\partial_j\left(\partial_i u\cdot\frac{w}{\rho} - N_i(w)\right) + \rho^3\partial_i\left(\partial_j u\cdot\frac{w}{\rho} - N_j(w)\right) - 2\rho^2\partial_{ij}u\cdot w + 2\rho^2 M_{ij}(w)
\end{aligned}
$$

□

Then, we can write the new version of Lemma 4.10:

LEMMA 4.14. *For u, v, p, w defined as in Lemma 4.13, for $f \in C^{2,\lambda}(B, \mathbb{R}^{n(n+2)/2})$, and for $1 \le i, j \le n$ with* $\operatorname{supp} f \subseteq B_1$, *if*

$$\partial_i u \cdot w = \rho N_i(w) \qquad \partial_{ij} u \cdot w = M_{ij}(w) - \frac{1}{2} f_{ij}$$

Then

$$\partial_i u \cdot \partial_j v + \partial_j u \cdot \partial_i v + \partial_i v \cdot \partial_j v = f_{ij}$$

Proof. This follows from substituting as we did in the proof of Lemma 4.14 and recognizing that $\rho(\operatorname{supp} f) = \{1\}$. □

The rest of the versions of Günther's lemmas and the resulting proof of Theorem 4.12 is essentially repetition from the previous section, so instead of proving everything (again), I include a brief summary of how to get the result.

For $v \in C^{s+2,\lambda}(B, \mathbb{R}^n)$, we can apply the boundedness of Δ^{-1} by Schauder and properties (8), (9), (10), (24), and (25) of Hölder norms to get estimates on N and M similar to those of Lemma 4.11, since N and M are linear combinations of dot products of derivatives of w of order no more than 2 multiplied by derivatives of ρ. Multiplication by ρ's derivatives changes nothing as every derivative of ρ bounded.

We will use the same pseudo-inverse $E(u)$ from before[27] and define $D(u)$ in the same way; (12) and Lemma 4.4 also continues to apply in the new case.

Finally, Theorem 4.12 can be proven by solving the fixed point problem via a similar iterative process:

$$w = E(u)\left(\rho N(w),\ -\frac{1}{2}f + M(w)\right)$$

Recall that what we are looking for is $v = \rho^2 w$.

As mentioned before, Theorem 4.12 can be applied to prove the following theorem:

THEOREM 4.15 ([27] Theorem 3). *Let (M, g) be a smooth Riemannian manifold, not necessarily compact. Let g be a C^∞ metric, $N \geq \frac{n(n+3)}{2} + 5$, $u_0 \in C^\infty(M, \mathbb{R}^N)$ such that the norms $\|\cdot\|_g$ and $\|\cdot\|_{g_{u_0}}$ induced by g and g_{u_0} respectively satisfy $\|\cdot\|_{g_{u_0}} < \|\cdot\|_g$, and $\delta \in C^0(M, \mathbb{R}^+)$. Then, there exists an embedding $u \in C^\infty(M, \mathbb{R}^N)$ such that $g_u = g$ and $\|u(p) - u_0(p)\| \leq \delta(p)$ for all $p \in M$.*

Recall that by Nash in [42] or Gromov–Rokhlin in 2.5.3 of [26], a free embedding is guaranteeed to exist if $N \geq \frac{n(n+5)}{2}$. Such a free embedding can be modified so that the condition on the induced metric for Theorem 4.15 is satisfied. Therefore, an immediate result from 2.5.3 in [26] and Theorem 4.15 is that any smooth Riemannian manifold can be smoothly isometrically embedded into Euclidean space of dimension

$$\max\left(\frac{n(n+5)}{2},\ \frac{n(n+3)}{n} + 5\right)$$

with the Euclidean metric.

A full proof of Theorem 4.15 in English is provided in Section 4 of [38] by Siyuan Lu.[28] It uses the following technique developed by Nash, which Andrews in [2] refers to as the *Nash twist*.

In [42, pp. 54–56], Nash showed that one can construct a sequence of C^∞ functions $f_1, \dots, f_{n(n+3)/2}$ on M such that the symmetric $(0,2)$-tensors defined by

$$\partial_i f_k \cdot \partial_j f_k \tag{33}$$

[27] The definition of $E(u)$ did not require M to be compact.

[28] The original proofs were given in German by Günther in [29]. In Section 4 of [38], Theorem 3.5 is Theorem 4.15.

are a generating set for the space of symmetric $(0,2)$-tensors at every point on M. By the condition $\|\cdot\|_{g_{u_0}} < \|\cdot\|_g$, $g - g_{u_0}$ can be written as a $C^\infty(M)$-linear combination of tensors of the form (33) with positive C^∞ coefficients a_k.

If we define $y_\varepsilon : M \to \mathbb{R}^{2m}$ (the "twist") by

$$y_\varepsilon^k := \frac{a_k}{\varepsilon}\sin(\varepsilon f_k)$$
$$y_\varepsilon^{m+k} := \frac{a_k}{\varepsilon}\cos(\varepsilon f_k)$$

The family of induced metrics

$$(g^\varepsilon)_{ij} := \sum_{k=1}^{m}\left(\partial_i y_\varepsilon^k \cdot \partial_j y_\varepsilon^k + \partial_i y_\varepsilon^{m+k}\cdot \partial_j y_\varepsilon^{m+k}\right) = \frac{1}{\varepsilon^2}\sum_{k=1}^{m}\partial_i a_k \cdot \partial_j a_k \tag{34}$$

can approximate $g - g_{u_0}$.[29]

Furthermore, one can choose local coordinates so that in such a neighborhood U_1, each term in the decomposition of $g - g_{u_0}$ has the form $a^4(dx_1 \otimes dx_1)$ (in other words, $f_k = x_1$) for some $a \in C^\infty(U_1)$ with compact support (so supp $h \subseteq U_1$). So the goal becomes to show that u_0 can be perturbed to a free embedding $\tilde{u}_0$ such that

$$\partial_i \tilde{u}_0 \cdot \partial_j \tilde{u}_0 = \partial_i u_0 \cdot \partial_j u_0 + a^4 \delta_{i1}\delta_{j1}$$

The coordinate neighborhood $U^{(1)}$ can be identified with the unit ball $B \in \mathbb{R}^n$ so that Theorem 4.12 can be applied to obtain the right perturbation. Theorem 3.13 in [38] is an alteration of Theorem 4.12 specifically aimed at finding this perturbation.

5. Consequences and Other Problems

In this section, I briefly introduce one major consequence of the John Nash's work in isometric embeddings: the Nash–Moser theorem. Then, I include some examples of other problems to which isometric embeddings apply.

5.1. *The Nash–Moser Theorem.* As mentioned previously, Nash's proof of his isometric embedding theorem, which involved constructing a non-standard inverse function theorem, turned out to be a very powerful result in analysis. There exists a generalization of Nash's proof into a new inverse / implicit function theorem on *Fréchet spaces*, which are complete Hausdorff metrizable locally convex topological vector spaces (For a further introduction to this

[29] The derivation of 34 is not difficult; it is included in Section 3.1 of Andrews's notes [2].

concept, see [30]). The first to produce an abstraction of Nash's results to Fréchet was Jacob Schwartz in [50] (published in 1960), but it was Jürgen Moser who first showed in [40] and [41] (published in 1966) that Nash's methods could be applied to other problems–in Moser's case, the nonlinear theory of positive symmetric systems in celestial mechanics. The result became known as the Nash-Moser theorem; below is Richard Hamilton's formulation in [30] (published in 1982):

THEOREM 5.1 (Nash–Moser). *Let $\mathcal{F}$ and $\mathcal{G}$ be tame Fréchet spaces and $P : \mathcal{F} \to \mathcal{G}$ a smooth tame map. Suppose $U \subseteq \mathcal{F}$ is an open set and, for all $f \in U$, $k \in \mathcal{G}$, the equations for the derivative $DP(f)h = k$ each have a unique solution $h = VP(f)k$ so that the family of inverses $VP : U \times \mathcal{G} \to \mathcal{F}$ is a smooth tame map. Then P is locally invertible, and each local inverse P^{-1} is a smooth tame map.*

The tameness condition on the spaces and mappings has to do with gradings on Fréchet spaces. [30] includes the full explanation of how calculus is done on Fréchet spaces as well as the proof of the Nash–Moser theorem (in particular, see Part II). Later on, Hamilton used the Nash–Moser result to prove the existence of Ricci flow in the 1980s, which eventually led to Perelman's proof of the Thurston geometrization conjecture (and therefore the Poincaré conjecture) in the early 2000s.

In the cases of both Riemannian isometric embeddings and Ricci flow however, mathematicians later discovered more elegant methods which avoided the monstrosities involved in proving the Nash–Moser result. As we saw in Section 4, in 1987, Günther did so and was able to improve the dimension of the ambient Euclidean space to $\max\left(\frac{n(n+3)}{2} + 5,\ \frac{n(n+5)}{2}\right)$.

5.2. *Isometric Embeddings of Surfaces into Euclidean Space.* Some interesting results have been developed specifically in the case of surfaces, which we explored a bit in Section 3.1. Mikhail Gromov was able to demonstrate that any compact surface can be isometrically embedded into $\mathbb{R}^5$ (see 3.2.4(D) in [25]).[30] Furthermore, any compact *orientable* surface can be isometrically embedded in $\mathbb{R}^4$ (see 3.2.4(C), in particular the exercise, in [25]). In Example 3.8, we discussed Greene's metric on $\mathbb{S}^2$ for which there is no C^2 isometric embedding into $\mathbb{R}^3$. Louis Nirenberg solved Weyl's problem in [43], however, showing that a 2-sphere with any C^4 metric inducing everywhere positive Gauss curvature can be isometrically embedded into $\mathbb{R}^3$. Alexei Pogorelov independently

[30]In comparison, Günther's proof allows surfaces to be isometrically embedded into $\mathbb{R}^{10}$.

proved something more general in [48]: A 2-sphere can be isometrically embedded in a 3-manifold whose curvature at every point is less than the lower bound for the Gaussian curvature induced by the metric on the 2-sphere at every point. For example, this means that any 2-sphere with a metric inducing Gaussian curvature greater than -1 can be isometrically embedded into hyperbolic 3-space.

5.3. *Isometric Embeddings in General Relativity.* In general relativity, we are concerned with studying solutions g to the Einstein equation

$$\text{Ric}g - \frac{1}{2}Sg = 8\pi T \tag{35}$$

where the unknown g–the gravitational field–is a Lorentzian metric,[31] S is the scalar curvature of g, and T is the stress-energy tensor, which encodes some type of "momentum density" information responsible for generating the gravitational field.[32] Of great importance in general relativity is the equivalence principle: Accelerating inertial reference frames–such as an elevator accelerating upward at 9.81...m/s–are indistinguishable from reference frames in gravitational fields–for example, standing still on Earth. The upshot is that gravity is not a force but an *effect* caused by the curvature of spacetime; the gravitational field is a *global* structure defined on the space-time manifold, and a mass moving around within a gravitational reference frame is not actually responsible for the curvature of space time. This phenomenon leads to ambiguity as to how to *locally* define mass density–that is, the problem of *quasi-local mass*. One may be able to find such information by doing an energy integral of T over some "space-like" 3-submanifold Ω describing a physical system. By conservation laws, expressed as $\nabla^\mu T_{\mu\nu} = 0$, this information depends only on the boundary $\partial\Omega$, a surface whose metric is induced by that of Ω. Therefore, we can look at problems of quasi-local mass from the point of view of isometric embeddings of surfaces.[33] For this problem, it is useful to consider surfaces that admit isometric embeddings into Minkowski space ($\mathbb{R}^{1,3} = \mathbb{R}^4$ with the metric $-\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$). Weyl's embedding theorem is useful here: See the work of Chiu-Chu Liu, Mu-Tao Wang, and Shing-Tung Yau in [36] and

[31] A Lorentzian metric g is similar to a Riemannian metric, but with signature $(-,+,+,+)$; in other words, at every point p, the matrix $g_{ij}(p)$ has one negative eigenvalue and three positive eigenvalues.

[32] One can think of this as a generalization of mass density in Newtonian mechanics and magnetic flux; the components $T^{\mu\nu}$ measures *momentum flux* (think momentum "flow") in the μ direction through a 3-submanifold describing a physical system whose normal is in the ν direction. The idea of a stress-energy tensor extends to other types of physical systems beyond the context of gravitation.

[33] For a more thorough explanation of the problem of quasi-local mass, see [59].

[60] to observe how this gets applied.

There is a specific case of the Einstein equation called the *Einstein vacuum equation*:

$$\mathrm{Ric} g = 0 \tag{36}$$

Solutions to (36) are called *gravitational instantons*. Maciej Dunajski and Paul Tod explored conformal and isometric embeddings of anti-self dual instantons[34] into Euclidean space in [15].

5.4. *Isometric Embeddings and Nonlinear Wave Systems.* The Einstein vacuum equation (36) is a type of *nonlinear wave system*. The use of isometric embedding results extend to other nonlinear wave systems as well. For example, Terence Tao has used Günther's solution to obtain results for solutions u to defocusing nonlinear wave systems of the form

$$-\partial_{tt} u + \sum_{i=1}^{d} \partial_{ii} u = \mathrm{grad}_{\mathbb{R}^m} F(u) \tag{37}$$

where $\mathbb{R}^{d+1}$ has the Minkowski metric and $\mathbb{R}^m$ has the Euclidean metric. In particular, Tao in [53] used Günther's solution to the isometric embedding problem to find that there exists a smooth potential F where Equation (37) admits solutions with finite-time singularities if the Euclidean dimension m is at least 76. He did this by deriving an isometric-embedding-type condition for u from the constraints of the system, where the stress energy tensor is the target "metric", and accordingly adapting the Nash embedding theorem.[35] In the zero-dimensional case, which is the well equation

$$-\partial_{tt} u = \mathrm{grad}_{\mathbb{R}^m} F(u) \tag{38}$$

Tao in [54] also used Günther's result to find solutions u that are global flows on some compact manifold. The idea is to find solutions to

$$\partial_t u = X(u)$$

the ODE for the global flow of a nonvanishing vector field X on a Riemannian manifold that push forward along an isometric embedding to solutions for (38).

[34] These are instantons g for which $\star g = -g$, where $\star$ is the Hodge operator, defined in Section 5.1 of [58].

[35] See Section 3 of [53] for more details.

5.5. *Isometric Embeddings and Calibrated Geometry.* Isometric embedding theorems also have an interesting relationship with calibrated geometry, the study of submanifolds of minimal volume, where the volume is restricted by a particular closed form (or *calibration*). One can think of this as an extension of minimal surface problems to higher dimensions. Calibrated submanifolds of interst include special Lagrangian submanifolds of Calabi–Yau (Ricci-flat Kähler) manifolds, associative and co-associative submanifolds of G_2-manifolds, and Cayley submanifolds of Spin(7)-manifolds (for a further introduction to these concepts, see [37]). Robert Bryant in [9] and Colleen Robles and Sema Selur in [49] proved the following:

(i) The interior of any closed, oriented, smooth 3-manifold can be isometrically embedded as an associative submanifold of a G_2-manifold. Furthermore, any closed, oriented, *analytic* 3-manifold can be isometrically embedded as a special Lagrangian submanifold of a Calabi–Yau 3-manifold.

(ii) The interior of any closed, oriented, smooth 4-manifold with trivial bundle of self-dual 2-forms can be isometrically embedded as a Cayley submanifold of a Spin(7)-manifold. Furthermore, any closed, oriented, *analytic* 4 manifold with trivial bundle of self-dual 2-forms can be embedded as a co-associative submanifold of a G_2-manifold.

Although $\mathbb{R}^7$ admits a G_2-structure, and $\mathbb{R}^8$ admits Calabi-Yau and Spin(7)-structures, the above does not imply that the generic 3- or 4-manifold with those specified conditions above can be isometrically embedded into $\mathbb{R}^7$ or $\mathbb{R}^8$.

5.6. *A Brief Note about Surface Matching.* Isometric embedding problems also have a place in surface matching problems in computer vision. In particular, the language of isometries and isometric embeddings applies to methods in comparing surfaces in the context of multidimensional scaling. See the work of Ron Kimmel and Asi Elad in [16] and Alexander Bronstein, Michael Bronstein, and Kimmel in [8] for more details.

6. **Reverberations**

We have spent this exposition discussing the *global* isometric embedding problem. Global embedding problems are present in just about every subfield of differential geometry and has also featured interactions with topology and algebraic geometry. In this exposition, we discussed global embeddings of Riemannian isometric embeddings that respect the Riemannian metric. In general, the goal of global embedding problems is to study embeddings that respect some special structure; in this section, we include (without proof) several results for the interested reader from symplectic geometry, contact geometry, complex geometry, and CR-geometry. A few of these results can be observed from the point of view of isometric embeddings.

6.1. *Holomorphic Embeddings*. Complex manifolds are those that locally resemble $\mathbb{C}^N$ and have holomorphic transition functions between coordinate charts. This forces the manifold to be orientable. A *holomorphic embedding* requires that the restriction of the coordinates of the ambient space to those of the manifold be holomorphic. Furthermore, a hermitian metric is defined in the same way as a Riemannian metric tensor, except that the inner product at each point is *hermitian* and is of the form $g = g_{j\bar{k}}\mathrm{d}z_j\mathrm{d}\bar{z}_k$. To each hermitian metric, we can associate the differential form $\omega = g_{j\bar{k}}\mathrm{d}z_j \wedge \mathrm{d}\bar{z}_k$. If $\mathrm{d}\omega = 0$, then the manifold is called *Kähler*, g is referred to as the *Kähler metric*, and ω is referred to as the *Kähler form*.

Kunihiko Kodaira's embedding theorem states that one can embed a compact Kähler manifold into some $\mathbb{CP}^n$ only if there exists a positive line bundle (see [4] and [5] for a proof by Donaldson). These can be realized as holomorphic isometric embeddings of the Kähler metric on the complex manifold into the Fubini–Study metric on $\mathbb{CP}^n$.[36]

Complex geometry and algebraic geometry have a great amount of intersection, and one often moves between the language of the two when studying complex varieties–that is, zero sets of homogeneous polynomials in complex variables. From the theory of algebraic curves, it is known that every compact Riemann surface (i.e. complex 1-manifold) can be algebraically–and therefore holomorphically, since an algebraic embedding is defined by polynomials–embedded in $\mathbb{CP}^3$ because of 1-1 correspondence between smooth projective algebraic curves and compact Riemann surfaces (see for example Corollary IV.3.6 and Theorem B.3.1 in [31]). By the theory of Weil divisors–in particular, the equivalence of Weil divisors and line bundles on algebraic curves–all Riemann surfaces meet the conditions for Kodaira's embedding theorem. By Chow's theorem, from which it follows that every complex submanifold of $\mathbb{CP}^N$ is an algebraic variety ([11] Theorem V), the Kähler manifolds that can be realized as *algebraic subvarieties* of $\mathbb{P}^n$–that is, the zero set of some collection of homogeneous multivariate polynomials in $n+1$ variables–are precisely the

[36] $\mathbb{S}^{2n+1}$ is not a complex manifold, but one can express the round metric in terms of complex coordinates by taking the pullback along inclusion of $\mathrm{d}z_k\mathrm{d}\bar{z}_k$ on $\mathbb{C}^{n+1}$. This metric induces another metric on the quotient $\mathbb{S}^{2n+1}/U(1) \cong \mathbb{CP}^n$. The induced metric on $\mathbb{CP}^n$ turns out to be Kähler and is known as the *Fubini–Study* metric.

ones that admit positive line bundles.

One cannot holomorphically embed a compact manifold into affine complex space $\mathbb{C}^n$ because such an embedding would have to simultaneously achieve maximum modulus and be open. However, Heinrich Behnke and Karl Stein proved that noncompact Riemann surfaces can be embedded holomorphically as submanifolds of affine complex space, or Stein manifolds (see Corollary 26.8 and Theorem 28.6 in [22]).

Isometric embedding problems are also studied in the context of Teichmüller theory. The *Teichmüller Space* of a Riemann surface S is the moduli space of complex structures.[37] Simultaneously, Teichmüller spaces can be viewed as the moduli space of hyperbolic metrics. This is related to the fact that most Riemann surfaces admit hyperbolic structures (i.e. metrics with constant sectional curvature -1) by the uniformization theorem. Teichmüller spaces themselves have the structure of a complex manifold, and studying their geometry is important in understanding the deformation theory on a Riemann surface. Some recent work has been done concerning isometric embeddings of Teichmüller spaces–see the work of Frederik Benirschke and Carlos A. Serván in [6], in which they were able to classify all isometric embeddings of Teichmüller spaces as branched coverings for complex dimension at least 2.

6.2. *Symplectic and Contact Embeddings.* Symplectic manifolds are even-dimensional manifolds with a non-degenerate closed 2-form; such manifolds are studied from the point of view of *area* as opposed to distance and curvature in Riemannian geometry. Taking powers of the symplectic form produces a nonvanishing volume form, so symplecticity also forces orientability. Contact manifolds are manifolds of dimension $2k+1$ that admit a smooth maximally non-integrable $2k$-distribution, or equivalently there exists a one form α such that $\alpha \wedge (\mathrm{d}\alpha)^n$ is nowhere vanishing.[38] Contact geometry is often viewed as the odd-dimensional analogue to symplectic geometry, though the translation is not always clean, as we will see in parts of this section.[39] [25] has chapters

[37]The existence of (holomorphic) complex structures is equivalent to, among other things, the existence of an automorphism J on the tangent bundle that satisfies $J^2 = -\mathrm{Id}$ and on which the *Nijenhuis tensor* of J, given below, is identically 0.

$$N_J(X,Y) = [X,Y] + J([JX,Y] + [X,JY]) - [JX,JY]$$

[38]$(\mathrm{d}\alpha)^n$ means $\mathrm{d}\alpha$ wedged n times.

[39]For instance, not all contact manifolds are orientable; see [12] for a discussion of nonorientable contact manifolds.

on global embeddings in symplectic and contact geometry. Under symplectic (resp. contact) embeddings, symplectic (resp. contact) form of the ambient space pulls back to the symplectic (resp. contact) form of the manifold.

In symplectic geometry, Gromov's non-squeezing theorem showed that a ball cannot be squeezed via symplectic embedding into a cylinder unless the radius of the sphere is at most the radius of the cylinder, which showed the strength of symplecticity as a condition (see Theorem 12.1.1 in [39]).

In [18] and [17], Yakov Eliasberg, Sang Seon Kim, and Leonid Polterovich studied the analogue of the nonsqueeze problem in contact geometry. The *prequantization space* $\mathbb{R}^{2n} \times \mathbb{S}^1$ has a contact structure, so one considers the problem of squeezing $B \times \mathbb{S}^1$ into $C \times \mathbb{S}^1$, where B is a ball and C is a cylinder. The results do not translate in general. There is a non-squeeze result if the difference between the radii of the cylinder and the ball is large enough, but results for when the difference is small are still to be determined at the time of writing. Where the difference between contact-squeezing and symplectic-squeezing is more apparent is in the case of two balls: If the radii are both smaller than 1, the balls *can* be contact-squeezed into each other; this is not true in the symplectic case.

Mentioned previously was Kodaira's embedding theorem of Kähler manifolds into complex projective space. Kähler manifolds are simultaneously complex and symplectic–in particular, the Kähler form is symplectic. For general symplectic manifolds (not necessarily complex), by Gromov and David Tischler, *every* symplectic manifold admits a symplectic isometric embedding into $\mathbb{CP}^N$, where the metric on the symplectic manifold is defined based on the symplectic form,[40] and the metric on $\mathbb{CP}^N$ is the Fubini–Study metric (see Theorem B in [55] or Exercise (1) under Theorem 3.4.2(A) in [25]).

Closed symplectic manifolds cannot be symplectically embedded into Euclidean space: All possible symplectic forms on Euclidean space are exact, which means that the symplectic form on the compact manifold would have to be exact. But this also means that the volume form obtained by taking a wedge power of the symplectic form must also be exact; by Stokes' theorem, the form integrates to 0, which contradicts the volume form being everywhere non-vanishing. On open (noncompact) symplectic manifolds, however, symplectic forms can be exact, and by results from Gromov concerning the h-principle,

[40]To see how this is done, see Proposition 5 in Section 1.3 of [32] and the preceding paragraph.

open manifolds with exact symplectic forms *always* admit embeddings into Euclidean space.[41]

Many closed *contact* manifolds can be contact-embedded into Euclidean space. Naohiko Kasuya in [34] and [33] proved that a closed co-oriented contact manifold M of dimension $2m+1$ can be contact-embedded into $\mathbb{R}^n$ in any of the following cases:

- $n = 4m+1$, $m \geq 3$ is odd, $H_1(M;\mathbb{Z}) = 0$, $\mathbb{R}^{4m+1}$
- $n = 4m+1$, $m \geq 4$ is even, and M is 2-connected
- $n = 4m+1$, $m = 2$ and M is simply connected
- $n = 5$, $m = 1$, and M has trivial first Chern class.[42]

For the first three cases, the contact structure on the ambient Euclidean space is the standard one; for the last case, a contact structure on $\mathbb{R}^5$ accepting a contact-embedding is proven to exist in [33]. Kasuya also proves in [33] that any codimension two closed contact submanifold of odd dimensional Euclidean space has trivial first Chern class. Therefore, a trivial first Chern class is a necessary and sufficient condition for a dimension 3 closed contact manifold to contact-embed into $\mathbb{R}^5$.

Simon Donaldson and Fabian Lehmann discovered a perturbation result pertaining to the problem of realizing closed 3-forms on 5-manifolds through embeddings into Calabi-Yau 3-folds in [14]. Integral to their approach was examining strongly pseudoconvex forms, which define contact structures. It is also explained that their problem is strongly related to that of CR-embeddings of CR-manifolds, introduced in the next section.

6.3. *CR-Embeddings.* A *Cauchy–Riemann (CR) structure* on a manifold M is a integrable[43] real subbundle of the complexified tangent bundle $TM_{\mathbb{C}} = TM \otimes_{\mathbb{R}} \mathbb{C}$ L such that $L \cap \bar{L} = 0$. A manifold with a CR-structure is referred to as a *CR-manifold*. A CR structure is a weaker form of a complex structure:

[41]In [19], see the description of the h-principle on page 54 and Theorem 7.2.3. by Gromov. One can also find the relevant concepts and results in [25]. h here stands for "homotopy"; The idea is that families that satisfy the h-principle satisfy some type of nice homotopy relation that allows one to, for example, approximate solutions to certain PDE's. Theorem 3.5 by Nash–Kuiper can be seen as a specific instance of the h-principle, for example.

[42]Chern classes are defined in Section 25.8 of [56].

[43]That is, closed under the Lie bracket

By Newlander–Nirenberg, if the dimension of M is even and L is of codimension 0, then M is a complex manifold.[44] As with the other geometries, a *CR-embedding* must be such that the CR-structure on the ambient space pulls back to the CR-structure on the manifold.

Global CR-embeddings into $\mathbb{C}^N$ are a very interesting subject. By the combined work of Louis Boutet de Monvel, Joseph Kohn, Hugo Rossi, Reese Harvey, and Blaine Lawson, we have that for compact strictly pseudoconvex CR-manifolds X, the following are equivalent:

- X is globally CR-embeddable into $\mathbb{C}^N$ for some N.
- X is the boundary of a normal Stein space (a generalization of Stein manifolds that allows singularities).
- The image of the Kohn Laplacian operator $\bar{\partial}_b$ over functions is closed.[45]

Stein spaces are even dimensional, so CR-manifolds admitting embeddings into $\mathbb{C}^n$ must be real odd dimensional.

Along with the above, we have by de Monvel that any compact strictly pseudoconvex CR-manifolds of dimension 5 and higher CR-embeds into $\mathbb{C}^n$. See [20] for more details.

For dimension 3, the Rossi spheres are counterexamples (see [1] for instance). But Sagun Chanillo, Hung-Lin Chiu, and Paul Yang proved in [10] if we additionally assume in dimension 3 that the CR Paenitz operator is non-negative and the scalar Tanaka–Webster curvature is positive, we have global CR-embeddability.[46]

There is a CR analogue of Kähler manifolds called *Sasakian manifolds*. Take a Riemannian manifold (M, g) and its *Riemannian cone* $M \times \mathbb{R}^+$ with metric $\bar{g} = t^2 g + \mathrm{d}t \otimes \mathrm{d}t$. M is Sasakian if $(M \times \mathbb{R}^+, \bar{g})$ is Kähler. Sasakian

[44] Mentioned in a previous footnote was yet another definition of a complex structure involving an automorphism J on the tangent bundle with $J^2 = 1$–called an *almost complex structure*–that is a full complex structure when the associated Nijenhuis tensor vanishes. These concepts are related in the following way: J yields a splitting of each complexified tangent space $T_p M_{\mathbb{C}}$ into eigenspaces: $T_p M_{\mathbb{C}} = T_p^{1,0} M \oplus T_p^{0,1} M$ where elements of $T_p^{1,0} M_{\mathbb{C}}$ (resp. $T_p^{0,1} M_{\mathbb{C}}$) are eigenvectors with eigenvalue i (resp. $-i$) under J. The distribution $T^{1,0} M = \bigcup_{p \in M} T_p^{1,0} M$ is a subbundle of $TM_{\mathbb{C}}$, and J is *integrable* (i.e. a *full* complex structure) when this distribution is integrable. See Chapter 2 of [58] for more details.

[45] For a further introduction to the Kohn Laplacian on CR-manifolds, see [45].

[46] See also [52] for a survey on the CR-Paenitz operator and its relation to global CR-embeddings.

manifolds are simultaneously contact and CR, where the CR-structure is induced by the contact structure. For a further explanation, see [7].

Liviu Ornea and Misha Verbitsky proved in [44] that any compact Sasakian manifold can be CR-embedded into another Sasakian manifold diffeomorphic to a sphere.

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Shiv Yajnik
Department of Mathematics, Columbia University
E-mail: columbiajournalofundergradmath@gmail.com
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