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One small step for stability, one giant leap for Schwarzschild: Boundedness of scalar waves on Schwarzschild spacetimes

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Abstract

This paper gives a detailed proof of the boundedness of the scalar wave equation on a Schwarzschild spacetime using modern energy estimates. The first proof of this statement was given by Kay and Wald in [10] by using Killing vector fields and discrete isometries of the spacetime. However, since then, new techniques to analyze vector fields near the event horizon have been developed by Dafermos and Rodnianski in [4] which has given rise to a new proof strategy. This new proof, using the red shift effect, has been sketched in many lecture notes (see [5], [3], and [9]), but many details are left out. This paper fills in those details and presents the theorem in a self-contained manner.

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1. **Introduction**

One of the major types of open problems in general relativity is the nonlinear stability of various black hole spacetimes. A *black hole spacetime* is one that has a complete future null infinity for which the past includes a regular future boundary. This regular future boundary is called the *event horizon*, and the future of the event horizon is the *black hole*. We can further define the entire past of null infinity to be the *black hole exterior*.

The simplest exact solution to the Einstein vacuum equation

$$\mathrm{Ric}(g) = 0 \tag{1}$$

that contains a black hole is the one parameter family of Schwarzschild metrics. These black hole spacetimes are spherically symmetric and static, and each metric depends on the parameter M which is interpreted as the mass of the black hole. In local coordinates (t, r, θ, ϕ), the metric take the form

$$g_M = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{2}$$

However in these coordinates, a Schwarzschild spacetime has two singularities: a metric singularity at $r = 0$ and a coordinate singularity at $r = 2M$. For this reason, only when Kruskal showed that the event horizon could be covered by regular coordinates, did the notion of a "black hole spacetime" become a formal definition [11]. This allowed for a geometric formulation of the question of stability. In his paper, Kruskal showed that we can consider new coordinates that take the form (T, R, θ, ϕ) where

$$T^2 - R^2 = \left(\frac{r}{2M} - 1\right) e^{r/2M} \qquad \text{and} \qquad 2M \log\left|\frac{T+R}{T-R}\right| = t.$$

In these coordinates and when $T^2 - R^2 < 1$, a Schwarzschild metric takes the form

$$g = \frac{32M^3}{r} e^{-r/2M}(-dT^2 + dR^2) + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right). \tag{3}$$

As it turns out, the one parameter family of Schwarzschild metrics is a subfamily of the two-parameter family of Kerr metrics. These metrics are stationary and axisymmetric and describe stationary rotating black holes. When the metrics are also spherically symmetric and static, they are called the Reissner-Nordström metrics, and take the form

(4)

$$g_{M,Q} = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 \left(d\theta^2 + \sin^2\theta \, d\phi^2\right).$$

Here, M is interpreted as the black hole mass and Q is interpreted as the electric charge of the black hole. For more information about the linear stability of Reissner-Nordström metrics, please see [8].

While the nonlinear stability of Kerr metrics is still an open problem, techniques have been developed through the study of simplified versions. This is exactly the case for uniform boundedness of scalar waves on Schwarzschild spacetimes—the main theorem of the present paper. The theorem was first proven in 1986 by Kay and Wald in [10].

Theorem 1.1. *Let* (M, g) *be a Schwarzschild spacetime and let* φ *be the smooth solution to the wave equation that satisfies some initial data* $f \in H^m_{\text{loc}}(\Sigma_0)$ *given on an initial data set* $(\Sigma_0, \not{g}, \kappa)$*. Then, there exists a* $C > 0$ *such that for all events* p *in exterior Schwarzschild region (including the horizons)*

$$|\varphi(p)| \leq C.$$

Their proof uses Killing vector fields and derivative estimates as outlined in Section 2 of this paper. However, it also relies heavily on discrete isometries of a Schwarzschild spacetime, making it difficult to generalize their methods in the nonlinear case. In 2005, Dafermos and Rodnianski developed methods to study non-Killing vector fields defined by the *local redshift effect* in Schwarzschild spacetimes. The redshift effect and surface gravity are discussed in Section 3. In fact, this construction generalizes Theorem 1.1 and has been useful for solving other problems in mathematical general relativity. This is discussed in more detail in Section 5.

The present paper uses the techniques developed by Dafermos and Rodnianski [4] to prove Kay and Wald's Theorem 1.1 in Section 4. The proof of this is outlined in many lecture notes (see [5], [3], and [9]), all of which leave out important details as exercises to the reader. This paper delves into those details, using rigorous proof techniques, to make the details accessible to newer students of the subject.

This paper is meant to be read by an advanced undergraduate student interested in learning more about general relativity. The ideal student has been introduced to the theory and has an advanced background in the analysis of partial differential equations. More specifically, she should have have knowledge about the following.

(1) With respect to general relativity, the reader is assumed to have studied Lorentzian geometry. She should know about causal properties and be able to analyze Penrose diagrams. Furthermore, she should know about trapped surfaces and black holes, the Penrose incompleteness theorem, and the Schwarzschild metric. Please refer to the first seven chapters of [1] for a full introduction to Lorentzian geometry.
(2) With respect to the analysis of partial differential equations, the reader should have a first year graduate student level background in theory. This means that she should have already seen the material outlined in the classic partial differential equations textbook by Evans [6].

2. **The Homogeneous Scalar Wave Equation**

Theorem 1.1 gives a pointwise bound for solutions of the homogeneous wave equation on Schwarzschild spacetimes. To understand what this entails, we will first define this equation on general Lorentzian manifolds. Let $(\mathcal{M}, g)$ be an $(n+1)$ dimensional Lorentzian manifold. For a scalar function $\varphi : \mathcal{M} \to \mathbb{R}$, the *covariant wave operator* $\Box_g$ is defined as follows.

$$\Box_g \varphi := \nabla^a \nabla_a \varphi = g^{\mu\nu} \partial_\mu \partial_\nu \varphi - g^{\mu\nu} \Gamma^\alpha_{\mu\nu} \partial_\alpha \varphi$$

Moreover, the *homogeneous wave equation* is

$$\Box_g \varphi = 0. \tag{5}$$

With respect to Minkowski spacetime $(\mathbb{R}^4, m)$ and in coordinates (t, x, y, z), this looks like

$$\Box_m \varphi = \partial_t^2 \varphi - \not{\Delta} \varphi = 0$$

where $\not{\Delta} \varphi$ represents the induced Laplacian of φ on $\mathbb{R}^3$.

We can view (5) as the Euler-Lagrange equation corresponding to the Lagrangian

$$\mathcal{L}[\varphi] = -\frac{1}{2} \sqrt{-g} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi$$

where $g = \det(g_{\mu\nu})$. Furthermore, the corresponding action to this density is

$$S[\varphi] = \int_{\mathcal{M}} \mathcal{L}[\varphi] \, d^n x = -\frac{1}{2} \int_{\mathcal{M}} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi \sqrt{-g} \, d^n x.$$

On the other hand, we can define the stress energy tensor $\mathbb{T}^\mu{}_\nu$ in terms of the same Lagrangian:

$$\mathbb{T}^\mu{}_\nu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}.$$

Definition 2.1. For a solution to the homogeneous wave equation φ, the *energy-momentum tensor* $\mathbb{T}_{\mu\nu}$ is

$$\mathbb{T}_{\mu\nu}[\varphi] = g_{\mu\alpha}\mathbb{T}^{\alpha}{}_{\nu}[\varphi] = g_{\mu\alpha}g_{\nu\beta}\mathbb{T}^{\alpha\beta}[\varphi] = \partial_\mu\phi\partial_\nu\varphi - \frac{1}{2}g_{\mu\nu}(g^{\alpha\beta}\partial_\alpha\varphi\partial_\beta\varphi).$$

In fact, for a solution to (5), the energy-momentum tensor is divergence free. To see this, one can easily derive that

$$\nabla^\mu\mathbb{T}_{\mu\nu} = (\Box_g\varphi)\nabla_\nu\varphi.$$

Thus, if $\Box_g\varphi = 0$, then $\nabla^\mu\mathbb{T}_{\mu\nu} = 0$. We call this property the *divergence identity* and it will play a very important roll in later calculations. Before that discussion, we need to understand another important property of the energy momentum tensor: the *positivity property.*

Theorem 2.2 (Positivity Property). *Let X and Y be two future directed causal vectors. Then,*

$$\mathbb{T}_{\mu\nu}X^\mu Y^\nu \geq 0. \tag{6}$$

Proof. First, note that the coordinate free formula for the energy momentum tensor is

$$\mathbb{T}(A,B) = (A\varphi)(B\varphi) - \frac{1}{2}g(A,B)|\nabla\varphi|^2$$

where A and B are two vectors. Now, define a null frame $(L, \underline{L}, e_1, e_2)$ such that

- L and $\underline{L}$ are future directed non-collinear null vectors normalizes so that $g(L,\underline{L}) = -2$,
- e_1 and e_2 are spacelike unit vectors orthonormal to L, $\underline{L}$, and to each other.

In this frame,

$$|\nabla\varphi|^2 = g^{\alpha\beta}\nabla_\alpha\varphi\nabla_\beta\varphi = -(L\varphi)(\underline{L}\varphi) + (e_1\varphi)^2 + (e_2\varphi)^2.$$

Since $g(L,L) = 0 = g(\underline{L},\underline{L})$ it follows that

$$\mathbb{T}(L,L) = (L\varphi)^2 \geq 0 \qquad \text{and} \qquad \mathbb{T}(\underline{L},\underline{L}) = (\underline{L}\varphi)^2 \geq 0.$$

On the other hand, the normalization $g(L,\underline{L}) = -2$ allows us to calculate

$$\begin{aligned}\mathbb{T}(L,\underline{L}) &= (L\varphi)(\underline{L}\varphi) - \frac{1}{2}(-2)\left(-(L\varphi)(\underline{L}\varphi) + (e_1\varphi)^2 + (e_2\varphi)^2\right)\\ &= (L\varphi)(\underline{L}\varphi) - (L\varphi)(\underline{L}\varphi) + (e_1\varphi)^2 + (e_2\varphi)^2\\ &= (e_1\varphi)^2 + (e_2\varphi)^2 \geq 0\\ &= \mathbb{T}(\underline{L},L).\end{aligned}$$

We can use this to calculate $\mathbb{T}(X,Y)$ when X and Y future directed causal vectors. If this is the case, then there exists some constants $a,b,c,d \in (0,\infty)$

such that

$$X = aL + b\underline{L} \qquad \text{and} \qquad Y = cL + d\underline{L}.$$

This means that

$$\begin{aligned}\mathbb{T}(X,Y) &= \mathbb{T}(aL + b\underline{L}, cL + d\underline{L}) \\ &= ac\,\mathbb{T}(L,L) + bc\,\mathbb{T}(\underline{L},L) + ad\,\mathbb{T}(L,\underline{L}) + bd\,\mathbb{T}(\underline{L},\underline{L}) \\ &= ac\,(L\varphi)^2 + bd\,(\underline{L}\varphi)^2 + (bc+ad)\left[(e_1\varphi)^2 + (e_2\varphi)^2\right] \geq 0.\end{aligned}$$

Therefore, $\mathbb{T}_{\mu\nu}X^\mu Y^\nu \geq 0$, which proves the positivity property. □

Remark 2.3. Note that this proof shows us that if X and Y are future directed timelike vectors, then

$$\mathbb{T}(X,Y) \geq c\sum_{i=0}^{n}(\partial_i\varphi)^2. \tag{7}$$

2.1. *Associated Energy Identity.* In the theory of hyperbolic partial differential equations, many times one will need to obtain estimates on solutions to such equations. This is sometimes done by integral identities. These are equations of the form

$$\int_{\partial\mathcal{R}} J \circ d\varphi = \int_{\mathcal{R}} K \circ d\varphi$$

where $\mathcal{R} \subset \mathcal{M}$ is some region and $\varphi : \mathcal{M} \to \mathbb{R}$ is a solution to the Euler-Lagrange equations associated to some Lagrangian $\mathcal{L}$. In general, we call sections J and K the *energy current* and *scalar current*, respectively. If this integral identity holds for all regions $\mathcal{R} \subseteq \mathcal{M}$ and for all solutions φ to the Euler-Lagrange equations, then we say that J is *compatible* with the Lagrangian $\mathcal{L}$. For more information about the theory of energy currents and hyperbolic partial differential equations, see [2].

Now in our setting, φ is a solution to (5) and we can precisely define the *energy current* J^X_μ and the *scalar current* K^X of the energy-momentum tensor such that

$$J^X_\mu[\varphi] := \mathbb{T}_{\mu\nu}[\varphi]X^\nu \qquad \text{and} \qquad K^X[\varphi] := {}^{(X)}\pi_{\mu\nu}\mathbb{T}^{\mu\nu}[\varphi]$$

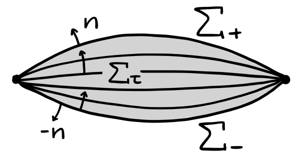
where X is a given vector field on $\mathcal{M}$. Here, ${}^{(X)}\pi^{\mu\nu}$ is the *deformation tensor* of X. Specifically we have that ${}^{(X)}\pi^{\mu\nu} = \frac{1}{2}(\mathcal{L}_X g)_{\mu\nu}$. In fact, we can define K^X in terms of J^X_μ. This is called the *divergence relation*:

$$K^X[\varphi] = \nabla^\mu J^X_\mu[\varphi]. \tag{8}$$

Note that since ${}^{(X)}\pi_{\mu\nu} = 0$ when X is Killing, it follows that

$$\nabla^\mu J^X_\mu[\varphi] = K^X[\varphi] = 0.$$

For this reason, when X is Killing, we call J^X_μ a *conserved current*.



Let us now suppose further that $(\mathcal{M}, g)$ is time oriented (meaning that it is a *spacetime*). Additionally, let $\mathcal{R} \subset \mathcal{M}$ be bounded by two homologous compact spacelike hypersurfaces Σ_+ and Σ_- that have a common boundary. Using Stokes' theorem and the divergence relation, we can define the following integral identity:

$$\int_{\partial\mathcal{R}} J^X_\mu[\varphi] n^\mu_{\partial\mathcal{R}} \, dVol_{g|\partial\mathcal{R}} = \int_{\mathcal{R}} K^X[\varphi] \, dVol_g$$

where $n^\mu_{\partial\mathcal{R}}$ is the future directed unit normal on the boundary of $\mathcal{R}$. Since $n^\mu_{\partial\mathcal{R}}$ is timelike (ie. $n_\mu n^\mu = -1$) and since the unit normal of $n^\mu_{\Sigma_-}$ is past directed, we get

$$\int_{\partial\mathcal{R}} J^X_\mu[\varphi] n^\mu_{\partial\mathcal{R}} = -\int_{\Sigma_+} J^X_\mu[\varphi] n^\mu_{\Sigma_+} + \int_{\Sigma_-} J^X_\mu[\varphi] n^\mu_{\Sigma_-}.$$

This means we have

$$\int_{\Sigma_+} J^X_\mu[\varphi] n^\mu_{\Sigma_+} = \int_{\Sigma_-} J^X_\mu[\varphi] n^\mu_{\Sigma_-} - \int_{\mathcal{R}} K^X[\varphi]. \tag{9}$$

Thus if X is a Killing vector field, then

$$\int_{\Sigma_+} J^X_\mu[\varphi] n^\mu = \int_{\Sigma_-} J^X_\mu[\varphi] n^\mu.$$

LEMMA 2.4. *Let $\mathcal{R} \subset \mathcal{M}$ such that its boundary is Σ_- and Σ_+, homologous spacelike hypersurfaces with common boundary. For a timelike vector field X, there exists a constant $C_{\mathcal{R}} > 0$ that only depends on $\mathcal{R}$ such that*

$$\|\varphi\|_{\mathring{H}^1(\Sigma_+)} + \|n\varphi\|_{L^2(\Sigma_+)} \leq C_{\mathcal{R}} \left(\|\varphi\|_{\mathring{H}^1(\Sigma_-)} + \|n\varphi\|_{L^2(\Sigma_-)} \right). \tag{10}$$

Remark 2.5. In this lemma, $\mathring{H}^1$ denotes the *homogeneous Sobolev space.* In the case of $p = 2$, we denote $D^{k,2}(\Omega)$ as $\mathring{H}^k(\Omega)$. This means that

$$\mathring{H}^k(\Omega) = \left\{ u \in L^1_{\mathrm{loc}}(\Omega) \;\middle|\; D^\ell u \in L^2(\Omega) \text{ for } |\ell| = k \right\} \quad \text{and} \quad \|u\|_{\mathring{H}^k(\Omega)} = \left(\sum_{|\ell|=k} \int_\Omega |D^\ell u|^2 \right)^{\frac{1}{2}}.$$

For more information about this space and others like it, please see [7].

Proof. Recall that for a spacelike hypersurface $\Sigma \subset \mathcal{M}$, we can define the following norms for functions f defined on Σ:

$$\|f\|_{\mathring{H}^1(\Sigma)} = \left(\int_\Sigma |\nabla\!\!\!/ f|^2 \right)^{1/2} \qquad \text{and} \qquad \|nf\|_{L^2(\Sigma)} = \left(\int_\Sigma |nf|^2 \right)^{1/2}.$$

Also, note that for a future directed timelike unit vector n, we can calculate

$$\begin{aligned}\mathbb{T}(n,n) &= (n\varphi)^2 - \frac{1}{2}g(n,n)|\nabla\varphi|^2 \\ &= (n\varphi)^2 - \frac{1}{2}(-1)\left(-(n\varphi)^2 + |\nabla\!\!\!/\varphi|^2\right) \\ &= \frac{1}{2}(n\varphi)^2 + \frac{1}{2}|\nabla\!\!\!/\varphi|^2.\end{aligned}$$

This shows us that we can study $|\nabla\!\!\!/\varphi|^2$ and $|n\varphi|^2$ by looking at its associated energy momentum tensor. Luckily, (9) will help us do just that.

To formalize this idea, we will start by defining a foliation of $\mathcal{R}$ so that

$$\mathcal{R} = \bigcup_{\tau\in[0,1]} \Sigma_\tau$$

where $\Sigma_0 = \Sigma_-$ and $\Sigma_1 = \Sigma_+$. Furthermore, denote $\mathcal{R}_\tau = \cup\Sigma_{\tau'}$ where $\tau' \leq \tau$. Assume that the interiors of Σ_τ are level sets of smooth functions $\phi = \tau$ on the interior of $\mathcal{R}$ such that for some constant $c > 0$, $-g(\nabla\phi, \nabla\phi) \geq c$. Assume that the interiors of Σ_τ are level sets of smooth functions $\phi = \tau$ on the interior of $\mathcal{R}$ such that for some constant $c > 0$, $-g(\nabla\phi, \nabla\phi) \geq c$.

For a timelike vector field X (to be specified later), applying (9) to the region $\mathcal{R}_\tau$ gives us

$$\int_{\Sigma_\tau} J^X_\mu[\varphi]n^\mu = \int_{\Sigma_-} J^X_\mu[\varphi]n^\mu - \int_{\mathcal{R}_\tau} K^X[\varphi].$$

Now, since both X and n are future directed timelike vector fields, the positivity property tells us that

$$c(x)\sum_{i=0}^{n}(\partial_i\varphi)^2 \leq \mathbb{T}(X,n) = J^X_\mu[\varphi]n^\mu$$

for some function $c(x) > 0$ that depends continuously on $x \in \Sigma_\tau$. On the other hand, since $K^X[\varphi]$ is defined to be the contraction of the energy momentum tensor with the deformation tensor, it is a linear combination of the squares of the first derivatives of φ. This means that there exists a continuous function $C(x)$ such that

$$|K^X[\varphi]| \leq C(x)\sum_{i=0}^{n}(\partial_i\varphi)^2.$$

But recall that $\mathcal{R}$ is compact. So by the Extreme Value Theorem, the continuous functions $c(X)$ and $C(X)$ on $\mathcal{R}$ attain maximums and minimums. Thus, there exists a constant $c_m > 0$ dependent on $\mathcal{R}$ such that $c_m \leq c(x)$, and there exists a constant $C_M > 0$ dependent on $\mathcal{R}$ such that $C(x) \leq C_M$. This shows us that

$$|K^X[\varphi]| \leq C_M \sum_{i=0}^{n} (\partial_i \varphi)^2 \leq \frac{C_M}{c_m} J^X_\mu[\varphi] n^\mu =: C J^X_\mu[\varphi] n^\mu.$$

We can use this inequality to bound $\int_{\Sigma_\tau} J^X_\mu[\varphi] n^\mu$. Specifically,

$$-\int_{\mathcal{R}_\tau} K^X[\varphi] \leq \int_{\mathcal{R}_\tau} |K^X[\varphi]| \leq C \int_{\mathcal{R}_\tau} J^X_\mu[\varphi] n^\mu.$$

Additionally, by the coarea formula, there exists some constant $c > 0$ dependent on $\mathcal{R}$ such that

$$dVol_{\mathcal{R}} \leq \frac{1}{c}\, d\tau\, dVol_{\Sigma_\tau}.$$

Putting these together,

$$C \int_{\mathcal{R}_\tau} J^X_\mu[\varphi] n^\mu \leq \frac{C}{c} \int_0^\tau \int_{\Sigma_{\tau^*}} J^X_\mu[\varphi] n^\mu.$$

This means that we have

$$\int_{\Sigma_\tau} J^X_\mu[\varphi] n^\mu \leq \int_{\Sigma_-} J^X_\mu[\varphi] n^\mu + \tilde{C} \int_0^\tau \int_{\Sigma_{\tau^*}} J^X_\mu[\varphi] n^\mu \tag{11}$$

where $\tilde{C}$ is a positive constant dependent on $\mathcal{R}$. Define a function f such that

$$f_X(\tau) := \int_{\Sigma_\tau} J^X_\mu[\varphi] n^\mu.$$

Rewriting (11) and remembering that $\Sigma_- = \Sigma_0$, we see

$$f_X(\tau) \leq f_X(0) + \tilde{C} \int_0^\tau f_X(\tau^*)\, d\tau^*.$$

The integral form of Grönwall's inequality tells us that the above inequality implies that

$$f_X(\tau) \leq f_X(0)\, e^{\tilde{C}\tau}$$

for every $\tau \in [0,1]$. Let $\tau = 1$, and define $\tilde{C}_{\mathcal{R}} = e^{\tilde{C}}$. Then

$$f_X(1) \leq \tilde{C}_{\mathcal{R}}\, f_X(0). \tag{12}$$

Now, observe that when $X = n$

$$f_n(\tau) = \int_{\Sigma_\tau} J^n_\mu[\varphi] n^\mu = \int_{\Sigma_\tau} T_{\mu\nu}[\varphi] n^\nu n^\mu = \int_{\Sigma_\tau} \frac{1}{2}(n\varphi)^2 + \int_{\Sigma_\tau} \frac{1}{2} |\nabla\!\!\!/ \varphi|^2 = \frac{1}{2} \|n\varphi\|^2_{L^2(\Sigma_\tau)} + \frac{1}{2} \|\varphi\|^2_{\mathring{H}^1(\Sigma_\tau)}.$$

This means that (12) states

$$\|n\varphi\|^2_{L^2(\Sigma_1)} + \|\varphi\|^2_{\mathring{H}^1(\Sigma_1)} \leq \tilde{C}_{\mathcal{R}} \left(\|n\varphi\|^2_{L^2(\Sigma_0)} + \|\varphi\|^2_{\mathring{H}^1(\Sigma_0)} \right).$$

To remove the squares, recall the standard algebraic inequality $\frac{1}{\sqrt{2}}(a+b) \leq \sqrt{a^2+b^2} \leq a+b$ where a and b are two positive constants. This means that

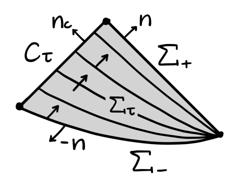
$$\frac{1}{\sqrt{2}}\left(||\varphi||_{\mathring{H}^1(\Sigma_1)} + ||n\varphi||_{L^2(\Sigma_1)}\right) \leq \sqrt{\tilde{C}_{\mathcal{R}}}\left(||\varphi||_{\mathring{H}^1(\Sigma_0)} + ||n\varphi||_{L^2(\Sigma_0)}\right).$$

Finally, define $C_{\mathcal{R}} = \sqrt{2\tilde{C}_{\mathcal{R}}}$. Then,

$$||\varphi||_{\mathring{H}^1(\Sigma_+)} + ||n\varphi||_{L^2(\Sigma_+)} \leq C_{\mathcal{R}}\left(||\varphi||_{\mathring{H}^1(\Sigma_-)} + ||n\varphi||_{L^2(\Sigma_-)}\right)$$

where we remember $\Sigma_+ = \Sigma_1$ and $\Sigma_- = \Sigma_0$. □

Now let us return to the situation when $(\mathcal{M}, g)$ is a Lorentzian manifold of dimension $n+1$, and $\mathcal{R}$ is a compact region. In this consideration, suppose that $\mathcal{R}$ is foliated by level sets Σ_τ of a smooth function $\phi = \tau$ such that $0 < c < |\nabla\phi| \leq C$. Suppose further that the foliation has a smooth null boundary C_τ. Geometrically, consider the figure below.



A difficulty in this scenario is that null cones do not come with a natural volume form nor a natural normal vector. However, if we choose the normal to be

$$n_C = -\partial_r$$

then one can calculate $g|_{C_\tau}$ to determine the induced volume form on C_τ. Stokes theorem then states

$$\int_{\Sigma_-} J^X_\mu[\varphi]n^\mu - \int_{\Sigma_+} J^X_\mu[\varphi]n^\mu - \int_{C_+} J^X_\mu[\varphi]n^\mu_C = \int_{\mathcal{R}} K^X[\varphi]. \tag{13}$$

The previous equality clearly implies the following inequality.

$$\int_{\Sigma_+} J^X_\mu[\varphi]n^\mu \leq \int_{\Sigma_-} J^X_\mu[\varphi]n^\mu - \int_{\mathcal{R}} K^X[\varphi]$$

So, we can define f_X so that

$$f_X(\tau) := \int_{\Sigma_\tau} J^X_\mu[\varphi]n^\mu.$$

Since we can still show that

$$\left|\int_{\mathcal{R}} K^X[\varphi]\right| \leq \int_0^\tau \int_{\Sigma_t} |J^X_\mu[\varphi]| d\mu_{\Sigma_t} dt \leq C\int_0^\tau f(t)dt$$

it follows from Grönwall's inequality that

$$f(\tau) \leq f(0)e^{C\tau}.$$

Using the same arguments as before, it's clear that the following theorem holds.

THEOREM 2.6. *Let $\mathcal{R} \subset \mathcal{M}$ such that its boundary is Σ_- and Σ_+ (homologous spacelike hypersurfaces) and C_+, a null hypersurface. For a timelike vector field X, there exists a constant $C_{\mathcal{R}} > 0$ that only depends on $\mathcal{R}$ such that*

$$\|\varphi\|_{\mathring{H}^1(\Sigma_+)} + \|n\varphi\|_{L^2(\Sigma_+)} \leq C_{\mathcal{R}} \left(\|\varphi\|_{\mathring{H}^1(\Sigma_-)} + \|n\varphi\|_{L^2(\Sigma_-)} \right). \tag{14}$$

2.2. *Well Posedness and Uniqueness.* While studying partial differential equations, questions about uniqueness and well posedness are natural to ask. It turns out, we can define uniqueness locally and within the domain of dependence for a given spacelike hypersurface. Furthermore, the wave equation on Lorentzian manifolds is well defined, resulting in well posedness.

COROLLARY 2.7 (Local Uniqueness). *Suppose that φ and $\tilde{\varphi}$ are two sufficiently regular solutions to (5) such that $\varphi = \tilde{\varphi}$ and $n\varphi = n\tilde{\varphi}$ on Σ_-. Then, $\varphi = \tilde{\varphi}$ in $\mathcal{R}$.*

Proof. Define $\psi = \varphi - \tilde{\varphi}$. Then, ψ is a sufficiently regular solution to (5) such that $\psi = 0$ and $n\psi = 0$ on Σ_-. Using the same foliation of $\mathcal{R}$ as given in the proof above, the lemma states that for each $\tau \in [0, 1]$,

$$||\psi||_{\mathring{H}^1(\Sigma_\tau)} + ||n\psi||_{L^2(\Sigma_\tau)} \leq C_{\mathcal{R}} \left(||\psi||_{\mathring{H}^1(\Sigma_-)} + ||n\psi||_{L^2(\Sigma_-)} \right) = C_{\mathcal{R}}(0) = 0.$$

Since $||\psi||_{\mathring{H}^1(\Sigma_\tau)}$ and $||n\psi||_{L^2(\Sigma_\tau)}$ are non-negative numbers, this inequality implies that

$$||\psi||_{\mathring{H}^1(\Sigma_\tau)} = 0 \qquad \text{and} \qquad \|n\psi||_{L^2(\Sigma_\tau)} = 0$$

on Σ_τ. Now using a Poincaré type inequality, we see that there is some constant $C > 0$ such that

$$\|\psi\|_{H^1(\Sigma_\tau)} \leq C \left(\|\psi\|_{\mathring{H}^1(\Sigma_\tau)} + \|\psi\|_{L^2(\partial\Sigma_-)} \right) = C(0) = 0. \tag{15}$$

Recall that $\partial\Sigma_- = \partial\Sigma_\tau$. Again, since $\|\psi\|_{H^1}(\Sigma_\tau)$ is non-negative, it follows that $\varphi = \tilde{\varphi}$ and $n\varphi = n\tilde{\varphi}$ on Σ_τ. Since this holds for every $\tau \in [0, 1]$, it holds in $\mathcal{R}$. ☐

THEOREM 2.8 (Uniqueness in the Domain of Dependence). *Let $(\mathcal{M}, g)$ be a 4-dimensional spacetime. Suppose that M has an acausal spacelike hypersurface $\Sigma \subset \mathcal{M}$ without a boundary. Let n be the future direction normal to Σ. If φ and ϕ are sufficient regular solutions to (5) such that*

$$\varphi|_\Sigma = \phi|_\Sigma \qquad \textit{and} \qquad n\varphi|_\Sigma = n\phi|_\Sigma$$

then $\varphi = \phi$ in $D(\Sigma)$ the domain of dependence of Σ.

Remark 2.9. The *domain of dependence* of Σ is the collection of points $p \in \mathcal{M}$ such that every inextendible causal curve through p also intersects Σ. This is exactly the *Cauchy development* of Σ. As the Cauchy development, $D(\Sigma)$ is globally hyperbolic and $\Sigma \subset D(\Sigma)$ is a Cauchy hypersurface. This means that if $\mathcal{M}$ is a Schwarzschild spacetime (and thus is globally hyperbolic), then $D(\Sigma) = \mathcal{M}$.

THEOREM 2.10 (Well Posedness). *Let $(\mathcal{M}, g)$ be a globally hyperbolic spacetime and let Σ be a Cauchy hypersurface of $\mathcal{M}$. Suppose that $\varphi \in H^s_{loc}(\Sigma)$ and $\varphi' \in H^{s-1}_{loc}(\Sigma)$. Then, there exists a unique ψ such that $\Box_g \psi = 0$ in $\mathcal{M}$ and for all spacelike hypersurfaces $\mathcal{S} \subset \mathcal{M}$,*

$$\psi|_{\mathcal{S}} \in H^s_{loc}(\mathcal{S}) \qquad n_{\mathcal{S}}\psi|_{\mathcal{S}} \in H^{s-1}_{loc}(\mathcal{S}) \qquad \psi|_{\Sigma} = \varphi \qquad n\psi|_{\Sigma} = \varphi'.$$

Remark 2.11. This theorem states that given some suitable initial data about an event in a Schwarzschild spacetime, there exists a unique solution to the wave equation (condition 1) that is “well behaved” (condition 2) and determines our initial data (condition 3). We will use the well posedness of the scalar wave equation as a general fact while proving its uniform boundedness of the Schwarzschild spacetime.

For the sake of length, the proofs of Theorem 2.8 and Theorem 2.10 are omitted from the paper. Please see [1] for a further explanation.

3. **The Redshift Effect**

An important result in mathematical general relativity was the analytic formalization of the *redshift effect*. In physics, “redshift” is a term usually used to discuss the wavelength of light increasing with respect to an observer while the object emitting the light is moving away. Many people know this redshift phenomenon as the Doppler effect, but it has many different applications throughout geometric optics. In fact in the early 2000s, this property made its way to the mathematical general relativity community, who at the time was in need of new techniques for studying non-Killing vector fields in spacetimes. This section will discuss the heuristics of the redshift effect and then will formalize the redshift dampening effect.

3.1. *Global and Local Redshift.* Consider the following thought experiment. There are two observers, A and B, near a black hole, and A is closer to the event horizon than B is. We will consider two different scenarios for the observers, and each one will lead to a different formalization of the redshift effect.

Global Redshift. Suppose that A crosses the event horizon and B does not. Furthermore right before A crosses, she sends a signal of constant frequency to B. B will receive this signal, but if she measures the frequency of it, she will find that it has “shifted to the red.” This phenomenon is called the *global redshift effect*, and it follows from the fact that the proper time of A before

crossing the event horizon is finite while the proper time of B is infinite. Its use in general relativity dates back to Oppenheimer and Snyder when they studied the appearance of a collapsing star to faraway observers in [12].
Local Redshift. Now, suppose that A crosses the event horizon and so does B but at a later time. Again, A sends a signal of constant frequency to B. This time, if B measures the frequency of the signal, she will find that it has "shifted to the red" by a factor depending exponentially on the difference in time that A and B crossed the horizon. This exponential factor is called the *surface gravity* of the event horizon, and its value determines the *local redshift effect*. This is the phenomenon Dafermos and Rodnianski formalize in [4] for the Schwarzschild spacetime geometry.

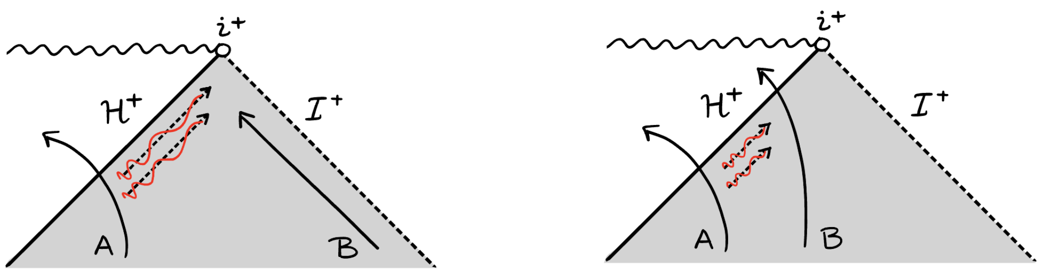


Figure 1. The figure on the left represents *global redshift* while the figure on the right represents *local redshift*.

3.2. *Redshift in Geometric Optics.* In order to formalize the redshift phenomenon, we first need to define the *surface gravity* of a black hole. This term will let us calculate the factor of dampening happening locally in the redshift effect.

Theorem 3.1 (Surface Gravity). *Let $(\mathcal{M}, g)$ be a black hole spacetime. Then, on the future event horizon of $\mathcal{M}$, there exists a constant $\kappa > 0$ and vector field T such that*

$$\nabla_T T = \kappa T.$$

Corollary 3.2 (Redshift Effect). *For a light signal sent at $t = 0$ from near the future event horizon of $\mathcal{M}$, the frequency of the signal is dampened by a factor of $e^{-\kappa t}$.*

The formalization of these statements was a major breakthrough in the mathematical study of general relativity. While they are vital to the construction of useful vector fields, the proofs are long and out of scope for the length of this paper. Please read [4] for a detailed discussion.

4. **Proof of Theorem 1.1**

Uniform boundedness for solutions to the scalar wave equation are achieved through the use of Sobolev inequalities for the solution's energy identities. Recall the main theorem from Section 1.

THEOREM 1.1. *Let* (M, g) *be a Schwarzschild spacetime and let* φ *be the smooth solution to the wave equation that satisfies some initial data* $f \in H^m_{\text{loc}}(\Sigma_0)$ *given on an initial data set* $(\Sigma_0, \not{g}, \kappa)$*. Then, there exists a* $C > 0$ *such that for all events* p *in exterior Schwarzschild region* (*including the horizons*)

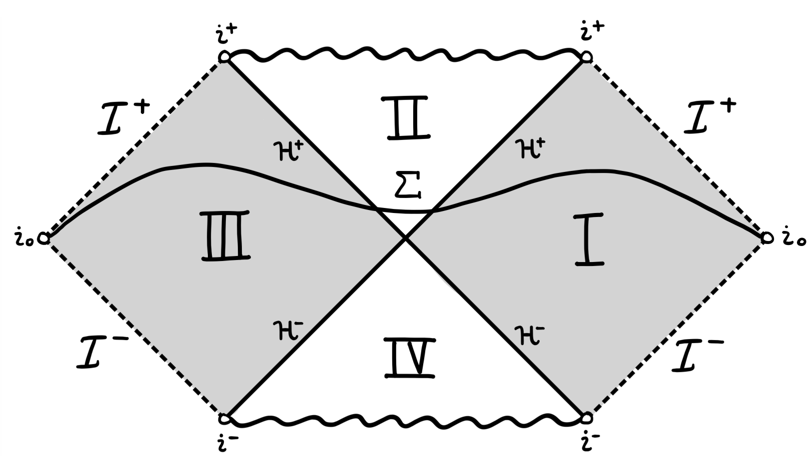
$$|\varphi(p)| \leq C.$$

The proof of this theorem is split into two main steps: pointwise bounds away from the event horizon and pointwise bounds including the event horizon. The first steps will use tools outlined in Sections 2 while the latter will require the non-Killing vector field discussed in Section 3 and a bootstrap argument. Before looking at these steps, however, we will first study the geometric nature of the problem.

4.1. *Schwarzschild Geometry.* Recall from Section 1 that the maximally extended Schwarzschild metric has Kruskal coordinates (T, R, θ, ϕ) and takes the form

$$g = \frac{32M^3}{r} e^{-r/2M} \left(-dT^2 + dR^2\right) + r^2 \left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

Furthermore, since this spacetime is spherically symmetric, $SO(3)$ acts by isometry. Thus, we can define $\mathcal{Q} = \mathcal{M}/SO(3)$ and the canonical projection $\pi : \mathcal{Q} \to \mathbb{R}^{1+1}$ such that the image of $\mathcal{Q}$ is compact in $\mathbb{R}^{1+1}$. This is the *Penrose diagram* for the maximally extended Schwarzschild metric.



In the diagram, Region I is the *exterior region*, and here $\{r > 2M, R > T\}$. This region is isometric to (2). Region II is the *black hole region* where $\{r < 2M, T < \sqrt{1+R^2}\}$. Region III is a *parallel universe*, or the second region isometric to (2) where $\{r > 2M, R > -T\}$. Finally, region IV is a *white hole* where $\{r < 2M, T > -\sqrt{1+R^2}\}$.

While the parallel universe and white hole regions sound like science fiction, it is important to note that they arise from the assumption that the spacetime is spherically symmetric. If we wish to generalize our methods to non spherically symmetric spacetimes, then we need to find a way to not include these regions. To do so, consider a different extended Schwarzschild coordinate system: one that covers Regions I and II separately from Regions III and IV. This is called the *Lemaitre coordinate system*, and Regions III and IV are interpreted as being the time reversal of Regions I and II.

Let (t, r, θ, ϕ) denote the standard Schwarzschild coordinates. Consider the change in coordinates where

$$t^* = t + 2M\log(r - 2M).$$

Then, on $r > 2M$ the metric in coordinates (t^*, r, θ, ϕ) takes the form

(16)

$$g = -\left(1 - \frac{2M}{r}\right)(dt^*)^2 + \frac{4M}{r}dt^* dr + \left(1 + \frac{2M}{r}\right)dr^2 + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right).$$

From this equation, we can calculate

$$g_{t^*t^*} = -\left(1 - \frac{2M}{r}\right) \qquad g_{t^*r} = \frac{2M}{r} \qquad g_{rr} = 1 + \frac{2M}{r} \qquad g_{AB} = \not{g}_{AB}$$

$$g^{t^*t^*} = -\left(1 + \frac{2M}{r}\right) \qquad g^{t^*r} = \frac{2M}{r} \qquad g^{rr} = 1 - \frac{2M}{r} \qquad g^{AB} = \not{g}^{AB}$$

The remaining metric components are zero. Additionally, we can express the covariant wave operator in this metric as

$$\Box_g \varphi = -\frac{1}{f^2}\partial_{t^*}{}^2\varphi + \frac{\nabla_i f}{f}\nabla^i\varphi + \not{\Delta}\varphi \qquad \text{where} \qquad f = \sqrt{1 - \frac{2M}{r}}. \tag{17}$$

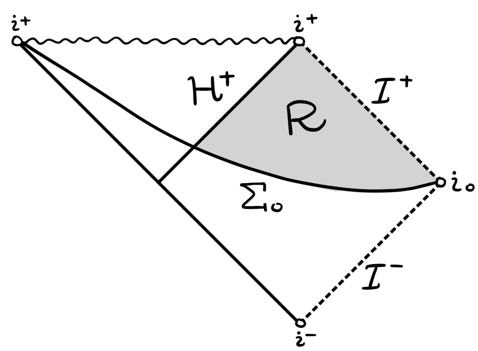
It's important to note that this coordinate system is well behaved when $r \geq 2M$ and it is isometric to Kruskal coordinates in Regions I and II. We will use these coordinates in the theorem, but we need to understand why we can restrict our analysis to Regions I and II.

First, the Schwarzschild spacetime $\mathcal{M}$ is spherically symmetric and Σ_0 is a Cauchy hypersurface, so we can assume that Σ_0 is also spherically symmetric. Additionally, observe that in the figure above, future null infinity $\mathcal{I}$ contains two disjoint regions: the future null infinity of Region I, $\mathcal{I}_I^+$, and the future null infinity of Region III, $\mathcal{I}_{III}^+$. This is also true for the past null infinity. This means that without loss of generality we can limit ourselves to the region

$$\mathcal{D} = \text{clos}(J^-(\mathcal{I}_I^+) \cap J^+(\mathcal{I}_I^-))$$

since one local Schwarzschild metric (2) covers this region. Since we will only refer to Region I for the remainder of the discussion, we will drop the subscript notation. However, if clarification is needed, it will be noted again. Since evolution of the wave equation is symmetric in time, we may also assume that Σ_0 does not intersect the past event horizon $\mathcal{H}^-$. Recall that the *past event horizon* is defined to be the past boundary of the future of the past null infinity.

We can further our limitations more. The wave equation is symmetric in time, so we can focus our study of the behavior of φ to $\mathcal{R} := J^-(\mathcal{I}^+) \cap J^+(\Sigma_0)$. Recall that the chronological past is bounded by the speed of light. Thus, the domain of dependence for this region is $\Sigma_0 \cap \mathcal{D}$, and φ is determined by $f|_{\Sigma_0 \cap \mathcal{D}}$. For this reason, it will suffice to prove the theorem assuming that Σ_0 is completely contained in exterior region and black hole region of the spacetime. Thus, we can use Lemaitre coordinates for our spacetime. The above reasoning means that we can consider the following figure.



Observe that we can equivalently define $\mathcal{R} = \{t^* \geq 0, r \geq 2M\}$. Let $T = \partial_{t^*}$. Since T is Killing, $K^T[\varphi] = 0$. Additionally, define

$$n = n^{t^*}\partial_{t^*} + n^r\partial_r \qquad \text{where} \qquad n^{t^*} = \sqrt{-g^{t^*t^*}} \quad \text{and} \quad n^r = -\frac{g^{t^*r}}{\sqrt{-g^{t^*t^*}}}.$$

This is a unit vector field. To see this, observe that

$$\begin{aligned} g(n,n) = g_{\mu\nu}n^\mu n^\nu &= g_{t^*t^*}(n^{t^*})^2 + 2g_{t^*r}n^{t^*}n^r + g_{rr}(n^r)^2 \\ &= -1 + \frac{4M^2}{r^2} - \frac{8M^2}{r^2} + \frac{4M^2}{r^2} \\ &= -1. \end{aligned}$$

Now, we can calculate

$$\begin{aligned}
g(T,n) &= g(T, n^{t^*}T + n^r\partial_r) \\
&= n^{t^*}g(T,T) + n^r g(T,\partial_r) \\
&= n^{t^*}\left(-1+\frac{2M}{r}\right) + n^r\left(\frac{2M}{r}\right) \\
&= -n^{t^*} + \frac{2M}{r}\left(n^{t^*}+n^r\right) \\
&= -\sqrt{1+\frac{2M}{r}} + \frac{2M}{r}\left(\sqrt{1+\frac{2M}{r}} - \frac{2M/r}{\sqrt{1+2M/r}}\right) \\
&= -\left(1+\frac{2M}{r}\right)^{-1/2}
\end{aligned}$$

If we let $f = 1 + (2M/r)$, then we can further calculate

(18)

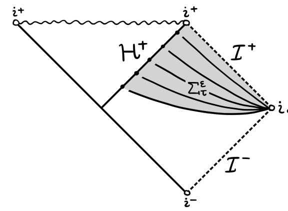
$$\begin{aligned}
J^T_\mu[\varphi]n^\mu &= \mathbb{T}(T,n) \\
&= (T\varphi)\left((n^{t^*}T + n^r R)\varphi\right) - \frac{1}{2}g(T,n)|\nabla\varphi|^2 \\
&= \sqrt{f}\,(\partial_{t^*}\varphi)^2 + \frac{1-f}{\sqrt{f}}(\partial_{t^*}\partial_r)(\varphi) + \frac{1}{2\sqrt{f}}|\nabla\varphi|^2 \\
&= \frac{1}{2}\left(\sqrt{f}\,(\partial_{t^*}\varphi)^2 + \frac{2-f}{\sqrt{f}}(\partial_r\varphi)^2 + \frac{1}{\sqrt{f}}|\nabla\!\!\!/\varphi|^2\right) \\
&= \frac{1}{2\sqrt{1+\frac{2M}{r}}}\left[\left(1+\frac{2M}{r}\right)(\partial_{t^*}\varphi)^2 + \left(1-\frac{2M}{r}\right)(\partial_r\varphi)^2 + |\nabla\!\!\!/\varphi|^2\right]
\end{aligned}$$

Since T is a time translation Killing field, we can define ϕ_τ to be the local flow of T. We assume that the hypersurface Σ_0 is the level set $\{t^* = 0\}$ of ϕ_τ. Since Schwarzschild spacetimes are globally hyperbolic, we can define spacelike hypersurfaces Σ_τ foliating $\mathcal{R}$ as the pushforward of Σ_0:

$$\mathcal{R} = \bigcup_{\tau\geq 0}\phi_\tau(\Sigma_0).$$

Additionally, T is timelike in $\mathcal{R}\backslash\mathcal{H}^+$ and it is lightlike on $\mathcal{H}^+$, making it a Killing horizon. Recall from Section 3 the black hole has surface gravity κ: specifically on $\mathcal{H}^+ = \{r = 2M\}$,

$$\kappa = \frac{1}{4M} \qquad \text{and} \qquad \nabla_T T = \kappa T \qquad \text{when} \qquad T = \partial_{t^*}.$$



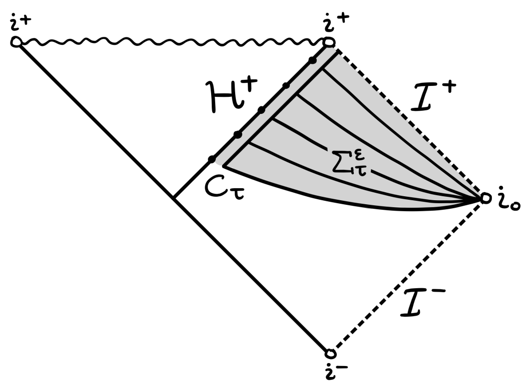
4.2. *Pointwise Bounds away from the Horizon.* We want to use Theorem 2.6, but it only holds when J^T is non-degenerate. The calculation in (18) shows us that this happens exactly at $r = 2M$. So, we will first prove Theorem 1.1 for hypersurfaces that are sufficiently disjoint from the event horizon. To construct these hypersurfaces, first let $S_\tau = \{t^* = \tau\} \cap \{r = 2M\}$ as the boundary of Σ_τ.

Note that S_τ is compact and that each Σ_τ is asymptopically flat. This means that there exists a compact $K_\tau = \{r \le R\} \cap \Sigma_\tau$ for R large such that $\Sigma_\tau \backslash K_\tau$ is covered globally by Euclidean coordinates. Let $\varepsilon > 0$, and denote

$$B^\varepsilon_0 = \Sigma_0 \cap \{r < 2M + \varepsilon\}$$

to be an ε neighborhood of S_0. Define $\Sigma^\varepsilon_0 := \Sigma_0 \backslash B^\varepsilon_0$. Then,

$$\Sigma^\varepsilon_\tau = \phi_\tau(\Sigma^\varepsilon_0) = \{t^* = \tau\} \cap \{r \ge 2M + \varepsilon\}.$$



Proposition 4.1 (Degenerate Energy Bounds). *For all $\varepsilon > 0$ there exists $C_\varepsilon > 0$ that only depends on ε such that for all τ and for all solutions φ of the homogeneous wave equation*

$$\sup_{\Sigma^\varepsilon_\tau} |\varphi| \le C_\varepsilon. \tag{19}$$

Proof. Consider $\mathcal{R}^\varepsilon_\tau \subset \mathcal{R}$ such that

$$\mathcal{R}^\varepsilon_\tau = \bigcup_{t^*=0}^{\tau} \Sigma^\varepsilon_{t^*}$$

Then, $\partial(\mathcal{R}^\varepsilon_\tau) = \Sigma^\varepsilon_\tau \cup C_\tau \cup \Sigma^\varepsilon_0$ where $C_\tau = \{r = 2M + \varepsilon, 0 \le t^* \le \tau\}$. Furthermore, T is Killing on $\mathcal{R}^\varepsilon_\tau$. This means that $K^T = 0$ and (13) states

$$\int_{\Sigma^\varepsilon_\tau} J^T_\mu[\varphi] n^\mu_{\Sigma^\varepsilon_\tau} + \int_{C_\tau} J^T_\mu[\varphi] n^\mu_{C_\tau} = \int_{\Sigma_0} J^T_\mu[\varphi] n^\mu_{\Sigma_0}. \tag{20}$$

Since T and each normal vector are all timelike on $\mathcal{R}^\varepsilon_\tau$, it follows from the positivity property that there is a constant $c \ge 0$ such that

$$J^T[\varphi]n \ge c\left((\partial_{t^*}\varphi)^2 + (\partial_r\varphi)^2 + |\nabla\!\!\!/\varphi|^2\right).$$

On the other hand, (20)

$$\int_{\Sigma^\varepsilon_\tau} J^T_\mu[\varphi] n^\mu_{\Sigma^\varepsilon_\tau} \le \int_{\Sigma_0} J^T_\mu[\varphi] n^\mu_{\Sigma_0}.$$

Putting these two inequalities together, it follows that

$$\|\varphi\|^2_{\mathring{H}^1(\Sigma^\varepsilon_\tau)} = \int_{\Sigma^\varepsilon_\tau} \left((\partial_{t^*}\varphi)^2 + (\partial_r\varphi)^2 + |\nabla\!\!\!/\varphi|^2\right) \le c_\varepsilon \int_{\Sigma_0} J^T_\mu[\varphi] n^\mu_{\Sigma_0} \le C_1(\varepsilon). \tag{21}$$

In the above, the last inequality holds since $\varphi \in H^m_{\text{loc}}(\Sigma_0)$ for a sufficiently large m. The value of $C_1(\varepsilon)$ only depends on ε since the integral is uniformly bounded and c_ε depends only on ε.

We also know that the function $T\varphi$ also satisfies the homogeneous wave equation, meaning that $\Box_g(T\varphi) = 0$, since T is Killing. Note that (21) holds for any solution to the homogeneous wave equation, so it follows that

$$\|T\varphi\|^2_{\mathring{H}^1(\Sigma^\varepsilon_\tau)} = \int_{\Sigma^\varepsilon_\tau} \left((\partial^2_{t^*}\varphi)^2 + (\partial_r\partial_{t^*}\varphi)^2 + |\nabla\!\!\!/\partial_{t^*}\varphi|^2\right) \le C_2(\varepsilon). \tag{22}$$

Recall that in a Schwarzschild spacetime, we can write the covariant wave operator as

$$\Box_g\varphi = -\frac{1}{f^2}{\partial_{t^*}}^2\varphi + \frac{\nabla_i f}{f}\nabla^i\varphi + \Delta\!\!\!/\varphi \qquad \text{where} \qquad f = \sqrt{1 - \frac{2M}{r}}.$$

Since $\Box_g\varphi = 0$, it follows that

$$\Delta\!\!\!/\varphi = \frac{1}{f^2}{\partial_{t^*}}^2\varphi - \frac{\nabla_i f}{f}\nabla^i\varphi.$$

Furthermore, since $r \ge 2M + \varepsilon$, f is uniformly bounded. This means that there exists two constants $D_1 > 0$ and $D_2 > 0$ such that

$$\|\Delta\!\!\!/\varphi\|_{L^2(\Sigma^\varepsilon_\tau)} \le D_1 \left\|{\partial_{t^*}}^2\varphi\right\|_{L^2(\Sigma^\varepsilon_\tau)} + D_2\|\nabla\varphi\|_{L^2(\Sigma^\varepsilon_\tau)}.$$

Using (21) and (22) we can further bound

$$\|\Delta\!\!\!/\varphi\|_{L^2(\Sigma^\varepsilon_\tau)} \le D_1C_2(\varepsilon) + D_2C_1(\varepsilon) =: C_3(\varepsilon).$$

By the homogeneous Sobolev embedding theorem, there exists a constant $D_3 > 0$ such that for each event $p \in \Sigma_\tau^\varepsilon$

$$
\begin{aligned}
|\varphi(p)| \leq \|\varphi\|_{L^\infty(\Sigma_\tau^\varepsilon)} &\leq D_3 \left(\|\nabla\varphi\|_{L^2(\Sigma_\tau^\varepsilon)} + \|\nabla^2\varphi\|_{L^2(\Sigma_\tau^\varepsilon)}\right) \\
&\leq D_4 \left(\|\nabla\varphi\|^2_{L^2(\Sigma_\tau^\varepsilon)} + \|\nabla^2\varphi\|^2_{L^2(\Sigma_\tau^\varepsilon)}\right) \\
&\leq D_4 \left(C_1(\varepsilon) + \|\partial_{t^*}{}^2\varphi\|^2_{L^2(\Sigma_\tau^\varepsilon)} + \|\not{\Delta}\varphi\|^2_{L^2(\Sigma_\tau^\varepsilon)}\right) \\
&\leq D_4 \left(C_1(\varepsilon) + C_2(\varepsilon) + C_3(\varepsilon)\right) =: C_\varepsilon.
\end{aligned}
$$

Note that the second inequality follows from Young's inequality while the remaining inequalities follow from (21) and (22). This proves our theorem. □

4.3. *Pointwise Bounds including the Horizon.* If we let $\varepsilon \to 0$ then $\Sigma_\tau^\varepsilon \to \Sigma_\tau$. However, as we noted before, there is a degeneracy in the energy current at $r = 2M$. Previously, we went around this by considering $\mathcal{R}_\tau^\varepsilon$, a region sufficiently disjoint from the event horizon. We now want to consider regions that include the event horizon. To do this, we need to construct a well behaved energy current of a non-Killing vector field. This is done through the use of the local redshift effect discussed in Section 3.

Proposition 4.2 (Local Redshift Vector Field). *There exists a vector field N such that*

(1) $(\phi_\tau)_* N = N$ *(meaning N is t^* independent)*
(2) *N is future directed timelike*
(3) *$N = T$ on $\Sigma_0 \backslash B$ for some compact B*
(4) *there exists an $\varepsilon > 0$ and $C > 0$ such that on $\Sigma_\tau \backslash \Sigma_\tau^\varepsilon$*

$$K^N[\varphi] \geq C J^N[\varphi] N \tag{23}$$

Proof. To prove this proposition, we will first prove that there exists a timelike vector field N_0 along $\Sigma_0 \backslash \Sigma_0^{2\varepsilon}$ such that (23) holds. Then, we will extend this vector field so that the proposition holds.

To construct such an N_0, consider T along S_0. Recall that $T = \partial_{t^*}$ is null here. Now, consider another null vector field

$$Y_0 = 2\partial_{t^*} - 2\partial_r$$

along S_0. This is in fact null since

$$g(Y_0, Y_0) = 4\left(g_{t^*t^*} - 2g_{rt^*} + g_{rr}\right) = 4(0) = 0.$$

Furthermore, observe the following computation.

$$g(T, Y_0) = 2g(\partial_{t^*}, \partial_{t^*}) - 2g(\partial_{t^*}, \partial_r) = 2\left(-1 + \frac{2M}{r}\right) - 2\left(\frac{2M}{r}\right) = -2$$

Now, we want to extend Y_0 off of the horizon to Y such that for some large $\sigma > 0$,

$$\nabla_Y Y = -\sigma(T+Y) \quad \text{on } S_0.$$

To do this, consider

$$Y = \left(2 + k_1\left(1 - \frac{2M}{r}\right)\right)\partial_{t^*} + \left(-2 + k_2\left(1 - \frac{2M}{r}\right)\right)\partial_r$$

where

$$k_1 = 1 + 3M\sigma \qquad \text{and} \qquad k_2 = 2 - 2M\sigma.$$

We now define N_0 as follows.

$$N_0 := T + Y = \left(3 + k_1\left(1 - \frac{2M}{r}\right)\right)\partial_{t^*} + \left(-2 + k_2\left(1 - \frac{2M}{r}\right)\right)\partial_r$$

We will now show that on $\Sigma_0 \backslash \Sigma_0^{2\varepsilon}$ there exists some constant $C > 0$ such that

$$K^{N_0}[\varphi] \geq C J^{N_0}[\varphi] N_0.$$

First note that since T is Killing, $K^T[\varphi] = 0$. So, $K^{N_0}[\varphi] = K^Y[\varphi]$. Second observe that

$$J^{N_0}[\varphi]N_0 = \mathbb{T}(N_0, N_0) = \mathbb{T}(T,T) + 2\mathbb{T}(T,Y) + \mathbb{T}(Y,Y).$$

Second, since T is Killing, $K^T[\varphi] = 0$. So, $K^{N_0}[\varphi] = K^Y[\varphi]$. Recall that $K^Y[\varphi] = {}^{(Y)}\pi_{\mu\nu}\mathbb{T}^{\mu\nu}$, and in local coordinates note that we can write

$${}^{(Y)}\pi_{\mu\nu} = D_\mu X_\nu + D_\nu X_\mu.$$

This allows us to preform the following calculation.

$$\begin{aligned}
K^Y[\varphi] &= {}^{(Y)}\pi_{\mu\nu}\mathbb{T}^{\mu\nu} \\
&= {}^{(Y)}\pi_{t^*t^*}\mathbb{T}^{t^*t^*} + 2^{(Y)}\pi_{t^*r}\mathbb{T}^{t^*r} + {}^{(Y)}\pi_{rr}\mathbb{T}^{rr} \\
&= (0)\mathbb{T}^{t^*t^*} + \frac{4Mk_1}{r^2}\mathbb{T}^{t^*r} + \frac{4Mk_2}{r^2}\mathbb{T}^{t^*r} \\
&= \frac{\kappa}{r}\left(k_1\mathbb{T}^{t^*r} + k_2\mathbb{T}^{rr}\right) \\
&\geq \frac{1}{4}\left(\mathbb{T}(Y,Y)\kappa + \mathbb{T}(T,Y)\sigma + \mathbb{T}(T,T)\sigma\right) \\
&\geq CJ^{N_0}(\varphi)N_0
\end{aligned}$$

We will now extend N_0. Let χ be the cutoff function such that in $\Sigma_0^{\varepsilon} \backslash \Sigma_0^{2\varepsilon}$, $\chi = 1$, and elsewhere $\chi = 0$. Then, since $\mathcal{I}^+$ is convex, define a future timelike vector field $\tilde{N_0}$ such that

$$\tilde{N_0} = \chi N_0 + (1-\chi)T.$$

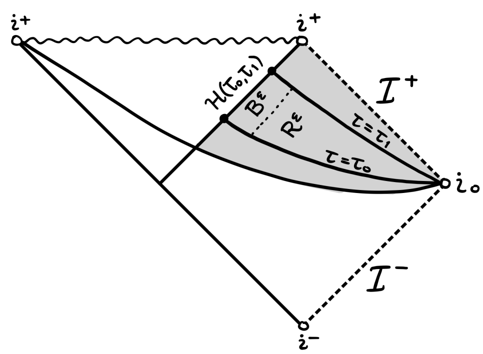
By definition, $\tilde{N_0}$ satisfies (23) on $\Sigma_0 \backslash \Sigma_0^{\varepsilon}$. Define $N = \phi_\tau(\tilde{N_0})$ to be the pushforward of $\tilde{N_0}$ along integral curves of T. Clearly, N satisfies the conditions of the proposition. $\square$

Proof of Theorem 1.1. Let $0 \leq \tau_0 < \tau_1 \leq \tau$. Denote $\Sigma_\tau = \Sigma(\tau)$. Consider the regions

$$\mathcal{R}(\tau_0,\tau_1) = \bigcup_{t^*=\tau_0}^{\tau_1} \Sigma(t^*) \qquad \text{and} \qquad \mathcal{H}(\tau_0,\tau_1) = \mathcal{H}^+ \cap \{\tau_0 \leq t^* \leq \tau_1\}.$$

Then for the region defined to the right, (13) states

$$\int_{\mathcal{H}(\tau_0,\tau_1)} J^N[\varphi]n + \int_{\Sigma(\tau_1)} J^N[\varphi]n + \int_{\mathcal{R}(\tau_0,\tau_1)} K^N[\varphi] = \int_{\Sigma(\tau_0)} J^N[\varphi]n \tag{24}$$



Recall that we can split $\mathcal{R}(\tau_0,\tau_1)$ into

$$R(\tau_0,\tau_1) = \bigcup_{t^*=\tau_0}^{\tau_1} (\Sigma(t^*)\backslash\Sigma^\varepsilon(t^*)) \;\;\cup\;\; \bigcup_{t^*=\tau_0}^{\tau_1} \Sigma^\varepsilon(t^*).$$

Denote $B^\varepsilon(\tau_0,\tau_1)$ as the first union and $\mathcal{R}^\varepsilon(\tau_0,\tau_1)$ as the second union. Then we can expand the integral over the interior region as

$$\int_{\mathcal{R}(\tau_0,\tau_1)} K^N(\varphi) = \int_{B^\varepsilon(\tau_0,\tau_1)} K^N[\varphi] + \int_{\mathcal{R}^\varepsilon(\tau_0,\tau_1)} K^N[\varphi]. \tag{25}$$

Combining the integral identities (24) and (25) gives us

$$\int_{\mathcal{H}(\tau_0,\tau_1)} J^N[\varphi]n + \int_{\Sigma(\tau_1)} J^N[\varphi]n + \int_{B^\varepsilon(\tau_0,\tau_1)} K^N[\varphi] = \int_{\Sigma(\tau_0)} J^N[\varphi]n - \int_{\mathcal{R}^\varepsilon(\tau_0,\tau_1)} K^N[\varphi].$$

Since the energy current is non-negative

$$\int_{\Sigma(\tau_1)} J^N[\varphi]n + \int_{B^\varepsilon(\tau_0,\tau_1)} K^N[\varphi] \leq \int_{\Sigma(\tau_0)} J^N[\varphi]n + \int_{\mathcal{R}^\varepsilon(\tau_0,\tau_1)} \left|K^N[\varphi]\right|. \tag{26}$$

Recall that the coarea formula states that for a real valued Lipschitz function $u : \mathbb{R}^n \to \mathbb{R}$ and domain $\Omega \subset \mathbb{R}^n$

$$\int_\Omega |\nabla u| = \int_{\mathbb{R}} H_{n-1}(u^{-1}(t))$$

where H_{n-1} is the $(n-1)$-dimensional Hausdorff measure. Applied to the energy currents of N, we have

$$\int_{B^\varepsilon(\tau_0,\tau_1)} K^N[\varphi] = \int_{\tau_0}^{\tau_1} \int_{B^\varepsilon(t^*)} J^N[\varphi]n \; dt^* d\tau \tag{27}$$

Then, by Proposition 4.2 there exists a vector field N such that for some $C_1 > 0$

$$C_1 \int_{\tau_0}^{\tau_1} \int_{B^\varepsilon(t^*)} J^N[\varphi]n \; dt^* d\tau \leq \int_{B^\varepsilon(\tau_0,\tau_1)} K^N[\varphi].$$

We can further bound this term from below by considering

$$-C_1 \int_{\tau_0}^{\tau_1} \int_{\Sigma(t^*)} J^N[\varphi]n \; dt^* d\tau \leq C_1 \int_{\tau_0}^{\tau_1} \int_{B^\varepsilon(t^*)} J^N[\varphi]n \; dt^* d\tau. \tag{28}$$

On the other hand, specifically on the complement of $B^\varepsilon(\tau_0, \tau_1)$, we have

$$\int_{\mathcal{R}^\varepsilon(\tau_0,\tau_1)} \left| K^N[\varphi] \right| \leq C_1 \int_{\tau_0}^{\tau_1} \left(\int_{\Sigma^\varepsilon(t^*)} J^N[\varphi]n \right).$$

Since $N = T$ on $\Sigma^{\varepsilon'}(0)$ for some $\varepsilon' > 0$, we have

$$\int_{\Sigma^\varepsilon(t^*)} J^N(\varphi)n \leq C_2 \int_{\Sigma(t^*)} J^T[\varphi]n \leq \int_{\Sigma(0)} J^T[\varphi]n =: D.$$

Note that the last inequality follows from analysis done in the proof of Proposition 4.1. So, putting all the above together, we derive

$$\int_{\mathcal{R}^\varepsilon(\tau_0,\tau_1)} \left| K^N[\varphi] \right| \leq C_1 C_2 (\tau_1 - \tau_0) D =: C_3 \tag{29}$$

Combining (26), (28), and (29) gives us

$$\int_{\Sigma(\tau_1)} J^N[\varphi]n - C_1 \int_{\tau_0}^{\tau_1} \int_{\Sigma(t^*)} J^N[\varphi]n \; dt^* d\tau \leq \int_{\Sigma(\tau_0)} J^N[\varphi]n + C_3. \tag{30}$$

Now define the function f by the following.

$$f(\tau) := \int_{\Sigma(\tau)} J^N(\varphi)n.$$

This means we can rewrite (30) as

$$f(\tau_1) - C_1 \int_{\tau_0}^{\tau_1} f(t^*) \leq f(\tau_0) + C_3.$$

Simple rearranging leads to

$$f(\tau_1) \leq f(\tau_0) + C_3 + C_1 \int_{\tau_0}^{\tau_1} f(t^*). \tag{31}$$

Notice that (31) holds for all $0 \leq \tau_0 < \tau_1 \leq \tau$. So, we can take $\tau_0 = 0$ and $\tau_1 = \tau$. Then,

$$f(\tau) \leq f(0) + C_3 + C_1 \int_0^\tau f(t^*) \tag{32}$$

By Grönwall's inequality

$$f(\tau) \leq f(0)e^{\tau C_4}.$$

Using the same argument as in Section 2, it follows that

$$||\varphi||_{\mathring{H}^1(\Sigma_\tau)} + ||n\varphi||_{L^2(\Sigma_\tau)} \leq C_5 \left(||\varphi||_{\mathring{H}^1(\Sigma_0)} + ||n\varphi||_{L^2(\Sigma_0)}\right). \tag{33}$$

for some constant $C_5 > 0$ independent of τ.

Recall that the Schwarzschild metric is spherically symmetric and asymptotically flat. This means that every hypersurface is also spherically symmetric and asymptotically flat. So by using a localized Sobolev embedding, we can find a constant $C_6 > 0$ such that for every $p \in \mathcal{R}$ and every $\tau \geq 0$

$$\sup_{\Sigma_\tau} |\varphi(p)| \leq C_6 \left(\|\varphi\|_{\mathring{H}^2(\Sigma_\tau)} + \|\varphi\|_{\mathring{H}^1(\Sigma_\tau)}\right) \tag{34}$$

Combining (33) and (34), we end up finding a constant $C > 0$ such that

$$\sup_{\Sigma_\tau} |\varphi(p)| \leq C_7 \left(||\varphi||_{\mathring{H}^1(\Sigma_0)} + ||n\varphi||_{L^2(\Sigma_0)}\right) \leq C.$$

The last inequality holds since, similar to the analysis for hypersurfaces separate from the event horizon, both of these norms are bounded by a constant depending only on the initial data set. □

5. Conclusion

As mentioned in Section 1, there is a natural generalization of Theorem 1.1. This was the theorem of focus in Dafermos and Rodnianski's original paper [4] where they formalized the local redshift effect. Their paper proves the following theorem.

Theorem 5.1 (Dafermos-Rodniaski, 2008). *Let ϕ be a sufficiently regular solution to the wave equation*

$$\Box_g \phi = 0$$

on the (maximally extended) Schwarzschild spacetime $(\mathcal{M}, g)$, decaying suitably at spatial infinity on an arbitrary complete asymptotically flat Cauchy surface Σ. Fix hindered and advanced Eddington-Finkelstein coordinates u and v on one of the exterior regions. For any achronal hypersurface $\mathcal{S}$ in the closure of this region, let $F(\mathcal{S})$ denote the flux of energy through $\mathcal{S}$, where energy is here measured with respect to the timelike Killing vector field. Let $v_+ = \max\{v, 1\}$,

$u_+ = \max\{u, 1\}$, *and* $v_+(\mathcal{S}) = \max\{\inf_{\mathcal{S}} v, 1\}$, $u_+(\mathcal{S}) = \max\{\inf_{\mathcal{S}} u, 1\}$. *We have*

$$F(\mathcal{S}) \leq C\left((v_+(\mathcal{S}))^{-2} + (u_+(\mathcal{S}))^{-2}\right).$$

(*We also allow* $\mathcal{S}$ *to be a subset of null infinity, interpreted in the obvious limiting sense.*) *In addition, we have the pointwise decay rates*

$$\begin{aligned} |\phi| &\leq C(v_+)^{-1} && \textit{in} \quad \overline{J^-(\mathcal{I}^+) \cap J^+(\mathcal{I}^-)} \\ |r\phi| &\leq C_{\hat{R}}(1+|u|)^{-1/2} && \textit{in} \quad \left\{r \leq \hat{R} < 2M\right\} \cap J^+(\Sigma). \end{aligned}$$

Although this result is still cited in present literature, 2008 was a long time ago, and there have been many advances to our understanding of stability in different black hole spacetimes. With respect to Schwarzschild spacetimes, linear stability was proven in Dafermos, Holzegel, and Rodnianski in 2016 (see [**?**]). This was an interim result between our Theorem 1.1 and the general nonlinear case. The general nonlinear case was solved in 2021 by Dafermos, Holzegel, Rodnianski, and Taylor when they proved the nonlinear asymptotic stability of solutions to the Einstein vacuum equation in the exterior region of the black hole (see [**?**]). They showed that generic vacuum initial data (meaning no assumed symmetry) sufficiently close to Schwarzschild data evolve to a vacuum spacetime that has the following properties:

(1) It possesses a complete future null infinity $\mathcal{I}^+$. The past of the future null infinity $J^-(\mathcal{I}^+)$ is bounded by a regular future complete event horizon $\mathcal{H}^+$.
(2) It remains close to Schwarzschild in the exterior region to the black hole.
(3) It asymptotically approaches a member of the Schwarzschild family as (an appropriate notion of) time goes to infinity.

The general nonlinear stability of Kerr spacetimes remains conjectured. However, Giorgi, Klainerman, and Szeftel proved nonlinear stability for small angular momentum in 2022 (see [**?**] and [**?**]). These proofs relied on the construction of the *general covariant modulation* (GCM) procedure which constructs a dynamical center of mass frame of the final state. An outline of this procedure can be found in Section 3 of [**?**]. Most recently, Fang, Giorgi, and Wan have developed a modification and extension of the GCM procedure in [**?**]. This extension is applicable to solutions of a broad class of matter models, such as the Kerr-Newman spacetimes which describe the geometry of rotating and electrically charged black holes. Current research is focused on proving the nonlinear stability of these spacetimes.

CONJECTURE 5.2. (*Nonlinear Stability of Kerr-Newman Spacetimes*) *Electrovacuum initial data sufficiently close to Kerr-Newman initial data have a maximal development with a complete future null infinity and a domain of outer communication which globally approaches a nearby Kerr-Newman solution.*

References

[1] Aretakis, S. Lecture notes on general relativity, 2013.

[2] Christodoulou, D. *The Action Principle and Partial Differential Equations.* (*AM*-146). Princeton University Press, 2000.

[3] Dafermos, M. The geometry and analysis of black hole spacetimes in general relativity, 2018.

[4] Dafermos, M., and Rodnianski, I. The red-shift effect and radiation decay on black hole spacetimes, 2005.

[5] Dafermos, M., and Rodnianski, I. Lectures on black holes and linear waves, 2008.

[6] Evans, L. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010.

[7] Galdi, G. P. *An Introduction to the Mathematical Theory of the Navier-Stokes Equations, Second Edition*. Springer, 2011.

[8] Giorgi, E. The linear stability of reissner–nordström spacetime: The full subextremal range $|q| < m$. *Communications in Mathematical Physics* 380, 3 (Nov. 2020), 1313–1360.

[9] Holzegel, G. Lecture notes vienna, 2014.

[10] Kay, B. S., and Wald, R. M. Linear stability of schwarzschild under perturbations which are non-vanishing on the bifurcation 2-sphere. *Classical and Quantum Gravity* 4, 4 (jul 1987), 893.

[11] Kruskal, M. D. Maximal extension of schwarzschild metric. *Physical Review* (1960).

[12] Oppenheimer, J. R., and Snyder, H. On continued gravitational contraction. *Phys. Rev.* 56 (Sep 1939), 455–459.

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