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Preservice Teachers' Reflections on the Use of Visual Supports to Improve Mathematics Pedagogy: A Case Study of Fraction and Ratio

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ABSTRACT This case study inspected mathematical learning opportunities that employed the results from the participating preservice teachers' (n=75) original visual arts creation process. Data obtained during this study provided empirical evidence for the educational viability of employing original Mondrian-style rectangle sets as a context for generating authentic mathematics inquiry opportunities. The participating preservice teachers generally indicated that through the process of creating their Mondrian-style rectangle sets and exploring the mathematical patterns within their works, such as the presence of fractions and ratios, they were able to develop an improved understanding of the pedagogy for teaching these mathematics concepts. The findings from this study suggest that both mathematics teachers and mathematics teacher educators might better serve their students by including more visual supports when teaching mathematics concepts.

KEYWORDS *mathematics education, teacher preparation, ratios and fractions, proportional reasoning, visual supports for instruction*

Introduction

The integration of visual representations such as tables, charts, symbols, diagrams, and graphs, into mathematics education has been a core pedagogical strategies since ancient times—the Pythagoreans used many of these techniques in their own self-edification and when teaching each other mathematics (Stylianou & Silver, 2004). Visual images used in mathematics can be generally divided into two main categories: (1) illustrative tools for organizing and communicating data; and (2) pedagogical tools for illuminating abstract and theoretical ideas via more tangible representations (Rellensmann et al., 2017). To facilitate the cognition of mathematics, building concept maps using either physical or mental images may help learners to create multifaceted connections between their new and existing knowledge (Friel, Curcio, & Bright, 2001). Visually perceived approaches to mathematics have the potential to facilitate students' mathematical comprehension, while also improving their spatial, proportional and operational execution (An, Cashman, & Tillman, 2021; An, Hachey, & Tillman, 2022).

As a pedagogical approach, visual literacy for mathematical concepts, in other words what occurs when information is illustrated through diagrams and charts, has shown wide applicability throughout multiple mathematical domains, including those embedded within the K-12 curriculum (Ainsworth, 2006). Classroom studies have verified that visual strategies for teaching mathematics generally outperform non-visual alternative strategies, particularly during problem solving tasks that require manipulation of both figures and quantities instead of just one or the other (Boaler et al., 2016). Pedagogically, visual literacy provides an accessible context for providing hands-on opportunities to develop mathematical knowledge, while also expanding the variety of contexts in which teachers can support the occurrence of such learning experiences (Honey et al., 2014; Guyotte et al., 2014).

Learning about mathematical ratios can provide a linkage among arithmetic, algebra and geometry—thereby integrating some of the most challenging instructional components within the elementary mathematics curriculum (Kilpatrick et al., 2001; National Mathematics Advisory Panel, 2008). Unfortunately, due to wide-ranging confusion about the various functions of mathematical ratios, there is a widespread misconception that mathematical ratios are too complex for teaching to early elementary students (Lamon, 2007; 2012). To address this misconception, and to facilitate elementary students' comprehension of mathematical ratios, teachers should offer students abundant opportunities to think proportionally while identifying, analyzing and representing fractional-proportional relationships (Fielding-Wells, Dole, & Makar, 2014). Compared with more commonly-used contexts that are often abstract in nature, such as math problems about speed variations or coupon discounts, visually-perceptible contexts (e.g. chorographical designs and graphic designs) have been used relatively infrequently for classroom illustration purposes (An et al., 2019; Weiland et al., 2021). Therefore, it seems worthwhile to examine potential approaches for developing visual supports into a teaching resource for exploring fractional relationships within elementary mathematics education.

During the past three decades, educational researchers and curriculum theorists have provided scholarship investigating visual supports as a means for improving mathematics education (e.g., Burton et al., 2000; Dana-Picard & Hershkovitz, 2019; Dietiker, 2015; Marshall, 2014; Nutov; 2021; Smithrim & Upitis, 2005). In general, this line of study has confirmed that positive effects on students' academic achievement can be achieved in situations where mathematics learning opportunities are visually supported. Within these studies, many aspects of the connections between the mathematics and visual supports have been articulated and assessed for pedagogical aptitude. However, a discovery from our literature review summarizing this field of study was that empirical research on the topic of mathematics education employing visual supports has as of yet not provided a replicable instructional model for assisting preservice teachers who are in the process of learning to use this pedagogical approach.

Nutov (2021) examined 127 preservice teachers on their learning experiences of learning the concepts of zero and infinity with the integration of visual art works in a mixed method study. The empirical evidence from Nutov's study suggests that mathematics education with visual supports positively correlates with preservice teacher motivation and positive feelings towards mathematics, as well as increased achievement in mathematical problem-solving. Within this context, the goal of this current study was to investigate preservice teachers' conceptual and pedagogical understanding of the relationship between visual supports, such as Mondrian-inspired artwork, and mathematical learning. The specific research question addressed by this research study was: how did the participating preservice elementary teachers evaluate their own mathematics pedagogy that resulted from learning about fractions and ratios with the help of visual supports?

Methods

Participants and Setting

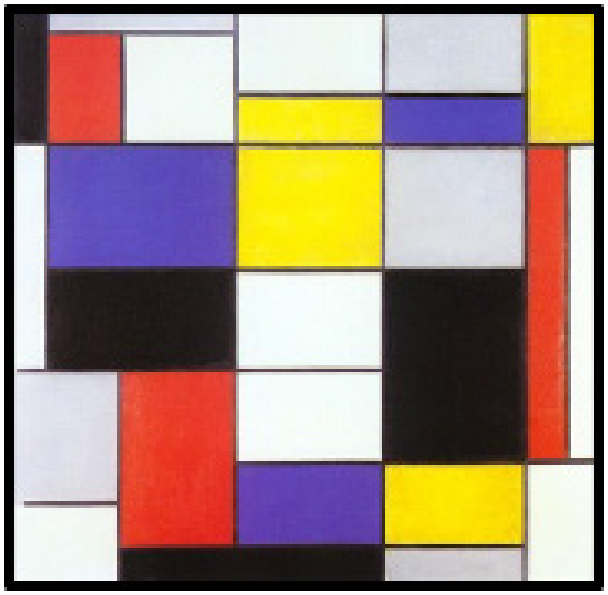
The participants in this case study were recruited from a research university along the southwestern border of the United States. As one of the largest Hispanic serving universities in the U.S., the recruitment location provided a population that was more than three-quarters Hispanic. A total of 75 preservice teachers were recruited from four different sections of an elementary mathematics methods course, all of which were taught by the same instructor and occurred over the duration of two regular academic semesters. The participating preservice teachers were senior-level undergraduates pursuing either an elementary generalist or an elementary bilingual generalist certificate, and a majority of them were working as student teachers in the local school systems. Among the 75 participants, 68 were female and 7 were male. Also, 71 of them self-identified as Hispanic and 4 of them self-identified as Caucasian.

Target Task and Procedure

During this study, the target task had three major components: (1) create an original piece of artwork consisting of a number of shaded shapes drawn in such a way to create a larger shape; (2) identify fractional values within the artwork for each shaded shape—each shaded area serving as a numerator and the whole area as the denominator; and (3) analyze the ratios of those fractional values—the ratio between the length and width

Figure 1

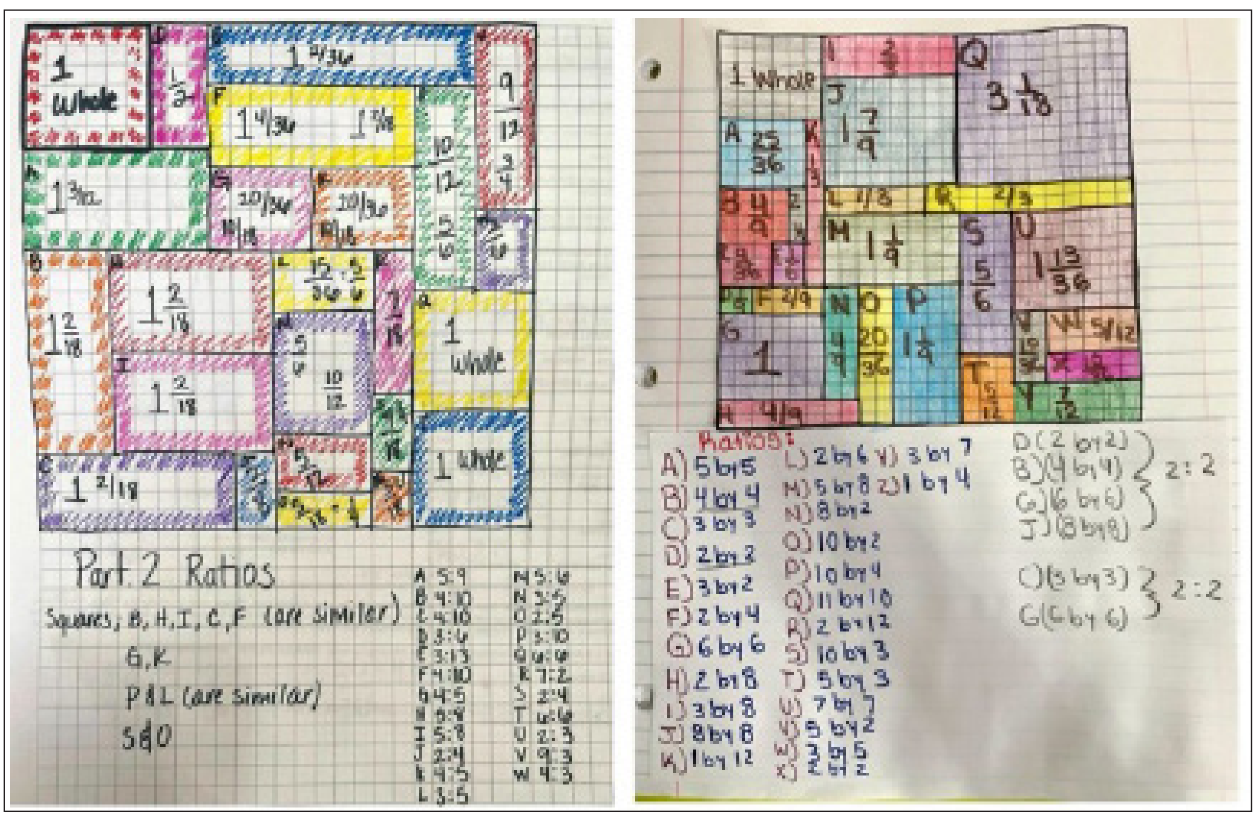
A painting titled "Composition A" (originally created in 1923, photo is public domain) by Piet Mondrian showing his highly geometrical style of art



within each shaded area. In the first component, all participants were taught about the painter Piet Mondrian (1872-1944) and the highly geometrical style employed in many pieces of his artworks (see Figure 1), after which they were guided through the process for creating their own original Mondrian-style rectangle sets on a sheet of letter-sized grid paper (see Figure 2). The definition of a single unit (i.e. *one whole* in fraction) was set as a 6×6 matrix with 36 tiny squares premarked on the top-upper corner of each sheet of grid paper that was distributed to the participants. To ensure sufficient complexity within the resulting artwork created by the participants so that they would have sufficient mathematical relationships to explore, the task directions required each participant to make at least five horizontal and five vertical lines in their artwork, thereby generating at least 25 rectangles. Next, participants were asked to determine the fractional value of each rectangle within their artwork by comparing each rectangle with the size of *one whole* unit as defined in the directions. Lastly, participants were asked to identify all

Figure 2

Example of a Mondrian-style rectangle set with corresponding fraction and ratio information created by two preservice teachers



of the geometrically similar rectangles, meaning those with equivalent ratios, that were present within their artwork.

Next, participants were tasked with identifying the correct fractional values of the rectangles within their original Mondrian-style designs. The predefined 6×6 square provided on the grid paper was used as a key reference by the participants as they computed each rectangle's fraction. The most commonly employed strategy was to construct a fraction by using the total number of grid squares as the numerator and 36 as the denominator. By using this method, rectangles that looked different but had the same areas nonetheless produced equivalent fractions. The emergence of mixed and improper fractions within the original Mondrian-style rectangle sets was an anticipated result from the first target task, because of the limited size of the provided grid paper. The emergence of equivalent ratios (based on part-to-part relationships between length and width) among each pair of geometrically similar rectangles from students' own artwork was an anticipated result from the second target task. The researchers observed that whenever the participants developed a rectangle that was larger than 36 grid squares, a fraction with a value larger than one whole would be formed. It should also be noted that while the participants each created their own original artworks based on the specified criteria provided in the task directions, they were also following a documentation process describing their experience and results. To demonstrate an understanding of fractions, the participants were directed to write down the fractional values presented within each rectangle on their artwork. To demonstrate understanding of ratios, the participants were asked to identify the ratio between width and length for each rectangle, and then write down all similar rectangles with equivalent length-to-width ratios on a separate piece of paper.

Data Collection and Analysis

Data collected in this case study included written essays, pictorial data in the form of original artwork, and hand-written documents for showing fraction and ratio attached to their artworks. To assist the data collection process, two online discussion forums were set up for participants so that they could upload their documents, as well as review each other's creations. The time span for the data collection was four days, which allowed each participant enough time to review the other's participants' artworks and their documented mathematical thinking. Specifically, first an online forum was created for participants to exhibit their original creations as well as share their corresponding mathematical analyses. Then, another online forum was provided three days later after the debriefing where each participant made additional comments regarding their own and each other's artwork and the researcher highlighted the two typical error patterns from existing art explorations (1) treating ratios as fractional values, and (2) employing ratios as if they were additive. In the first type of error pattern, some participants mistakenly treated the fractional values of each individual rectangle's area as interchangeable with the width to length ratios of similar rectangles. In the second type of error pattern, some participants have misconceptions that a rectangle with a width to length ratio of 2:3 has an identical ratio to a rectangle with a width to length ratio of 3:4 because the case is increased by 1 unit which is a "smooth" proportional change. This second forum was used by participants to share post-task reflection essays describing their thoughts on this pedagogical approach for teaching mathematics to elementary students.

The collected data were analyzed utilizing a grounded theory approach (Corbin & Strauss, 2008). Specifically, the collected pictorial data and corresponding written data were paired up and analyzed together as a means for investigating any potential linkages between the participants' original artwork designs, mathematical problem-solving strategies, and the contents of their pedagogical reflections. A content analysis was undertaken with the primary goal of pinpointing and describing the distinguishing methods, strategies, and judgments performed across the entire set of data from the 75 participants. During the first round of coding, individual emergent themes from the collected data were clarified with the aim of generating a broad-spectrum understanding of the participant's general reasoning processes along with identifying any misconceptions about fractions and ratios. This task was accomplished through constant comparisons until the initial entry categories were qualitatively saturated. During the second phase, the data were refined by escalating quotations worthy of being reported, including those pertaining to interesting problem-solving strategies and noteworthy pedagogical reflections; this phase also included sharpening the refinement of the selected representational cases while also eliminating repeated entries with similar content. To minimize any unconscious coding biases, two separate coders worked independently throughout the data analysis process and the results were collectively compared to determine confirmation, or the lack thereof, for the results that emerged. The inter-rater agreement rate was about 93% and the 7% emergent inconsistencies in the coding results were then resolved through open discussion conducted among the research team members until a consensus was achieved.

Results

The participating preservice teachers collectively indicated that by creating their own Mondrian-style rectangular shape sets, they were guided to explore the mathematical patterns such as fraction and ratio within their produced works. They were able to develop an improved understanding of fractions and ratios, as well as the pedagogy appropriate for teaching these two mathematics concepts. A total of 14 themes emerged (see Table 1) after analyzing the collected data and these themes were classified into four major categories, which included: (1) participant identified math content knowledge challenges (e.g., clarified the differences between fraction and ratio); (2) participant developed deepened conceptual understanding of specific math topics (e.g., illustrated parts-to-whole relationship); (3) participant identified the pedagogical connections among math topics (e.g., taught fractions with connections to ratios); and (4) participant generated a positive learning environment (e.g., improved engagement in mathematical investigations). To illustrate the pedagogical reflections in detail, seven participants' testimonies—Evelyn, Joanna, Sabrina, Paula, and Yasmin (the names of the selected participants have been changed to pseudonyms)—were selected to emphasize their distinctive perspectives.

By manipulating a set of rectangles with different length and width characteristics, the participants were provided an opportunity to constructively identify the patterns displayed between the emergent fractions and corresponding ratios. This process appears to have enhanced the participants' self-reported knowledge for teaching mathematics. As an illustration, Evelyn shared personal insights into her own challenges understanding the conceptual differences between fractions and ratios:

> After completing this activity, I realized out of all my educational years, even now that I am all ready to finish my degree in university, I never really worked with ratios with a hands-on activity before and was never taught the difference between a ratio and a fraction. I found that this interactive visual activity helped me reason and be able to picture each rectangle and the relation it had to the whole. Although

Table 1

Themes from Participants' Reflections on Creating Mondrian-Style Rectangle Sets

Major Theme	Specific Theme	Count / Rate (n=75)	
Identified the math content knowledge challenges	Identify flawed understanding of proportions	42	56.0%
	Clarify the differences between fractions and ratios	31	41.3%
	Differentiate additive and multiplicative relationships	7	9.3%
Deepened conceptual understanding of specific math topics	Conceptualize parts-to-whole relationship	39	52.0%
	Conceptualize equivalent fractions concept	31	41.3%
	Conceptualize parts-to-parts relationship	25	33.3%
	Conceptualize improper fractions concept	14	18.7%
Identified the pedagogical connections among math topics	Teach fractions with connections to ratios	70	93.3%
	Teach ratios with connections to geometry/measurement	29	38.7%
	Teach fractions with connections to geometry/measurement	23	30.7%
	Teach ratios with connections to algebras	4	5.3%
Created positive learning environments	Improve engagement in mathematical investigations	34	45.3%
	Embed math knowledge within fun tasks	28	37.3%
	Flexibly adjust difficulty for different grade-levels	11	14.7%

> both fraction and ratio have the concept of part and whole, but ratio is not restricted to the part-to-whole relationship all the time, it may display a part-to-part relationship (e.g., length vs width) as we explored in similar rectangles.

Likewise, another participant named Joanna shared her insights about the pedagogical potential from creating original Mondrian-style artwork as a method for learning about differences between fractions and ratios. Joanna stated that drawing multiple rectangles simultaneously within her original Mondrian-style artwork allowed her to see the ratios emerging among the different rectangles from an empirical perspective. For her, the most challenging part of the ratios-identification task was the rule against making any conclusions based merely on appearance, as she found the requirement of providing mathematical evidence to be quite an unfamiliar approach compared to her traditional, visual method for addressing these types of problems. Joanna explained this peculiar scenario in her own words:

> Even though I consider myself to be pretty good with fractions, I was stunned by observing so many assorted rectangles in this artwork. I was able to see dozens of fractional relationships across all rectangles because they came directly from a whole picture that I could constantly compare and that engaged me with a clear perspective of what I was working. In a ratio sense, I think it is difficult because our eyes can be very deceiving. Especially for the rectangles that are slightly longer or shorter in their length/width, it was hard to tell whether their ratios are equivalent or not because our eyes can't catch these nuances. I need to mathematically analyze the rectangles using the [grid] squares inside and relate them to each other by determining an even scale at which the rectangle either proportionally increases or decreases.

Many participants stated that Mondrian-style rectangle sets offered opportunities to pedagogically connect mathematics topics, especially between number and geometry. Sabrina described her reflections about the pedagogical value of the visual supports employed, and how during her experience the Mondrian-style rectangle sets empowered her to uncover methods for explaining the mechanics of how fraction works, while also linking concepts about fractions with concepts pertaining to geometry and proportion. Sabrina remarked:

> When the concepts are presented separately, fractions are commonly associated only with numerical equations. Now when using Mondrian designs, teaching fractions can be transformed with connection to geometric and proportional reasoning. I think that manually manipulating rectangles in the Mondrian designs helps students have a better understanding and more practice while thinking about all the possible sizes, number of squares and portions they get. If my future student finds the concept of fractions confusing, I can guide them do the basic counting of squares to obtain an answer. Now, in reference to the prompt visual representations, my students will have the freedom to explore and make connections with equivalent fractions as well as how and why to simplify fractions.

In comparison with the traditional hands-on activities that generally feature combining parts into a whole and then decomposing that whole back into parts, participants described how the Mondrian-style rectangle sets enabled them to move beyond individual examples of fractions and toward an understanding of the larger abstract concept of fractions. The learning featured activities representing and comparing many fractions simultaneously, and this was one of the distinctive elements that facilitated students to explore fractional relationships. Moreover, the creative process of making Mondrian-style original artworks created an engaging learning environment that may improve students interest in mathematics. One of the participants, Paula described her insights about the mathematics teaching method examined:

> Many hands-on activities like cutting and pasting as well as manipulatives for representing fractions do not give the students a whole representation. Mondrian designs were systematic, it was easy to follow and complete. I would like to say that this is the first time I do something like this, with my own hands. It enabled me to acknowledge the advanced ways that multiple fractions can be represented at the same time and how each fraction is related to the other fractions. It would be much easier for students to first activate their artistic motivation by setting up an enjoyable learning environment, and then to develop their math and logical thinking by connecting art and math. This is where I can see how to guide my students to build their foundational knowledge

> of fraction, a concept which I myself had a negative disposition before but now became more interesting and engaging.

As did Paula, a number of participants reported that their experiences completing the Mondrian-style task improved their awareness of limitations to traditional fraction-teaching approaches, especially from their own past learning experiences. For example, instead of replicating how she was taught as an elementary student, Yasmin said that she is going to change her pedagogy for instructing fractions to her future students. Yasmin noted:

> Many teachers use one and only one representation (i.e., pizza) to teach fraction and [it] can cause future comprehensive conflicts when other teaching approaches are used. I remember fraction being taught straight to the point with just one process to follow to get to the answer. My teachers used to teach us how to conduct fraction operations so we used to memorize the routine process but we never understood the why or the reason behind that. I never had the opportunity to develop critical thinking by exploring through hands-on activities like the Mondrian Arts activity. The Mondrian Arts activity helped me to realize that the same fraction can have different representations, yet their space occupied will always be the same.

Discussion

This research study empirically examined innovative pedagogy that emphasized participants creating and using original visual artworks to support the learning of mathematics. The investigated mathematical tasks were selected to allow the researchers an opportunity to better understand the instructional potential for visually-based fractional and proportional representations during mathematics teaching. Our findings were consistent with previous studies (Buforn & Fernández 2014; Hilton & Hilton 2018; Livy & Vale 2011) that showed preservice teachers were commonly underprepared for instructing their students on the topics of fractions and ratios; likewise, most of the participants in this study self-identified the presence of personal knowledge challenges when analyzing their own understanding of fractions and ratios. Results from this study revealed that when a judiciously combined fractional task and proportional task were provided, many of the participants were able to improve their mathematics content knowledge by transforming previously disconnected concepts into interrelated topics.

Connections between ratios and the fractional values among the rectangles the participants created in their Mondrian-style rectangle sets enabled instructional insights. Specifically, during the first task a part-to-whole relationship was the primary focus during an area comparison contrasting each of the individual rectangles and the pre-defined reference rectangle encompassing six grid squares. However, in the second task, a part-to-part relationship was the major focus during a comparison of the lengths and widths within each individual rectangle. Results from this study showed that most of the participants employed visual-spatial strategies, such as comparing a predetermined "one whole" rectangle that was provided to help them formulate and check their fractional reasoning as they created Mondrian-style rectangles. Consistent with several previous studies (Buforn & Fernández, 2014; Hilton & Hilton, 2018; Livy & Vale, 2011), the results also demonstrated that a substantial percentage of the participating preservice teachers were aware of the limitations to their own mathematics knowledge in regard to understanding fractions and ratios. Post-activity reflections from many of the preservice teachers indicated that the visual supports during the instruction helped them to identify their own mathematical misunderstandings. Thus, in line with Nutov (2021), our findings also indicate that mathematics instruction, especially when introducing concepts with dissimilar symbolic representation but equivalent mathematical values (i.e. improper fractions and mixed fractions), can be effectively supplemented by the appropriate use of visual supports as the bridge to consolidate the knowledge connections.

During data analysis, the researchers noted that the participants' pedagogical knowledge for teaching fractions was often overly reliant on the use of mathematical procedures. This finding was not surprising, since procedural fluency is often a main focus of K-12 math instruction, but this almost exclusive focus on procedure appears to have resulted in the impairment of other, deeper and more impactful approaches to engaging with math topics (Ostler, 2011). As our study's findings illustrated, some of the participants learned how to make connections between their visual perceptions and the abstract mathematical concepts they will be responsible for teaching—and these results provide evidence that mathematics instruction might be more effective if visual supports are systematically employed where appropriate. Overall, the findings from this study suggest that elementary teacher educators and elementary teachers may best serve their students by mixing visual supports with their mathematics pedagogy.

Both national and state-level mathematics teaching standards have historically recommended exposing students to transdisciplinary learning opportunities such as finding connections between mathematics and the arts, as was done in this research project wherein mathematical concepts were embodied through a visually-based format (National Council of Teachers of Mathematics, 2000, 2006). This study highlighted several central issues impacting the effectiveness of teacher education related to the teaching and learning of fractions and ratios. The empirical findings from this study demonstrated that preservice teachers could benefit from additional preparation at using contextualized pedagogical approaches to generate high-quality instruction for teaching fractions and ratios effectively. This approach would also help to remedy any gaps in preservice teachers' content knowledge about fractions and how they are related with ratios and proportions.

Mondrian-style rectangle sets have distinctively embedded geometrical and proportional components, which can offer classroom teachers a convenient and comprehensive teaching resource for enabling students to explore complex proportional relationships within fractions and ratios by using visual supports. Generating Mondrian-style rectangle sets by directing the creative placement of horizontal and vertical lines can result in a mathematical learning resource that effectively utilizes the art creation and appreciation processes. The exceptionally structured and geometrical nature of Mondrian-style art can enable a lush pedagogical context for investigating the differences and similarities between fractions and ratios, assisting both students and their teachers (Weiland et al., 2021). So that the instructional approach described in this study might become inclusive of teaching other mathematical topics in addition to ratios and fractions, the current paper concludes by encouraging further research systematically examining the pedagogical potential and challenges pertaining to the use of visual supports as an effective supplement for teaching K-12 mathematics.

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