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NOTES FROM THE FIELD

A Problem-based Curriculum to Develop the Multiplication Principle for Counting

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Introduction

Multiplication is relatively simple and is at the core of solving counting exercises. The multiplication principle (MP) specifies the conditions under which we should use multiplication in combinatorics problems. Yet counting exercises are notoriously difficult (e.g., Batanero et al., 1997; Hadar & Hadass, 1981). Given this difficulty, we explore nuances of the MP for counting that address students' difficulties. This paper proposes a problem-based curricular sequence to develop the MP for counting, with a focus on problem-based learning in combinatorics education.

Problem-based Learning and Combinatorics Education

Problem-based learning (PBL) is a pedagogy that is organized around problem-solving activities. PBL facilitates meaning-making, constructing, discovering, and engaging with information with a structured sequence of tasks (Rhem, 1998). Recent studies suggest that PBL is an effective pedagogical approach to mathematics education, increasing students' interest and motivation (Cerezo, 2004; Mahmood & Jacobo, 2019).

Research suggests that students hold misconceptions about multiplication in combinatorics (Lockwood et al., 2015). For this paper, we use Tucker's (2012) definition of the MP (italics added):

> "The Multiplication Principle: Suppose a procedure can be broken into m successive (*ordered*) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m *different* outcomes in the mth stage. If the number of outcomes at each stage is *independent* of the choices in previous stages and if the *composite outcomes are all distinct*, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes" (p. 180).

This definition captures the conditions under which multiplication can be used in counting. However, research suggests that students grapple with significant challenges in the context of counting problems (e.g., Lockwood & Purdy, 2019).

The following four collections of PBL exercises highlight the subtleties of the MP in combinatorics by providing pedagogical scaffolds through the structure and sequence of these exercises.

Collection 1

One way we can learn is through contrast, that is, understanding concept x by contrasting it with concept y. Table 1 highlights the fundamental difference between multiplication and addition through contrast of these concepts, albeit in the same context of the exercises.

In order to solve Exercise 1, multiplication must be used in a two-stage process. For example, we might first pick a donut and then a coffee. The resulting breakfasts are composite outcomes—i.e., (donut, coffee) pairs. For each of the five types of donuts, we might choose any of four flavors of coffee, giving us a total of 20 possible breakfasts. In general, with d options for donuts and c options for coffee, $d*c$ (donut, coffee) pairs are possible. Exercise 1 might be used to illustrate the process-oriented approach to counting used by the MP, in which we use ordered stages to perform a procedure that results in composite outcomes.

Table 1

Exercises 1-2

Exercise 1: At your local café are 5 types of donuts and 4 flavors of coffee. You decide to order a breakfast consisting of a donut and a cup of coffee. How many different possible breakfasts could you order?
Answer: 5*4=20.

Exercise 2: At your local café are 5 types of donuts and 4 flavors of coffee. You decide to order a breakfast consisting of exactly one item. How many different possible breakfasts could you order?
Answer: 5+4=9.

Addition must be used to solve Exercise 2. In contrast to the breakfasts in Exercise 1, the breakfasts in Exercise 2 are *not* the composite result of a two-stage process but instead are selections from two disjoint sets—breakfasts consisting of a donut and breakfasts consisting of a coffee. Counting, however, still might take place in two stages. First, we count the five donut breakfasts, and then we count the four coffee breakfasts for a total of 5+4=9 breakfasts. In general, with d options for donuts and c options for coffee, $d+c$ breakfasts are possible. These problems highlight the counting principles to distinguish multiplication from addition while providing the same contextual elements.

Collection 2

A second way we learn is through generalization, a process through which students experience sameness by observing that a concept applies across a range of relevantly similar instances. But this brings up the question of how those instances are similar. Exercises 3-5 in Table 2 introduce several real-world scenarios for counting problems—a café, routes between towns, and a class of students—that can be used to generalize important facets of multiplication in counting exercises.

Compared to Exercise 1, Exercise 3 introduces a new stage to the counting process: coffee with or without cream. Thus, *for each* of the 5*4 breakfasts in Exercise 1, you can now order your coffee in one of two ways—with or without cream. This notion that "for each outcome," there are x more options is a critical way to understand the MP. Breakfasts are now (donut, coffee, cream) triples—the result of a three-stage counting process that yields 5*4*2 composite outcomes. Exercise 3 also introduces a dichotomous option (with or without) that is often useful in counting.

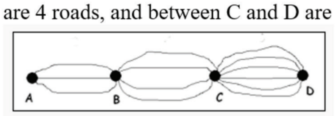
Exercise 4 provides a new context but uses the MP in much the same way as in Exercise 3. Like Exercise 3, part (i) of Exercise 4 requires an approach to counting that uses the MP with three successive ordered stages. This time, though, we have three options in the first stage, four in the second, and six in the third. Hence, the solution is 3*4*6. Parts (ii) and (iii) extend the context, adding even more stages to the process. Part (ii) introduces a modality to the experience in which, *for each* of the 3*4*6 total routes, you could travel in one of three ways—driving, biking, or walking. In part (ii), you cannot switch from one mode of transportation to another—say, driving from town A to town B, biking from town B to town C, and walking from town C to town D. The travel modality must remain consistent for the entire journey. Each composite outcome expands with the modality option, and experiences are now (road A, road B, road C, mode) quadruples. Unlike part (ii), part (iii) allows you to switch from one mode of transportation to another at the towns. For example, if you drive from town A to town B, you can still either drive, bike, or walk from town B to town C. As a result, the distinct composite outcomes change from quadruples to sextuples: experiences are (road A, mode A, road B, mode B, road C, mode

Table 2

Exercises 3-5

Exercise 3: At your local café are 5 types of donuts and 4 flavors of coffee. Coffee is served either with cream or without. You decide to order a breakfast that consists of a donut and a coffee. How many different possible breakfasts could you order?
Answer: 5*4*2.

Exercise 4: Consider towns A, B, C, and D. Between A and B are 3 roads, between B and C are 4 roads, and between C and D are six roads, as shown below.



(i) How many routes are there to get from A to D?
(ii) Suppose you are either going to drive, bike, or walk. How many different experiences are there to get from A to D?
(iii) Suppose on each road you could drive, bike, or walk. How many different experiences are there to get from A to D?
Answers: (i) 3*4*6. (ii) 3*(3*4*6); (iii) (3*3)*(4*3)*(6*3).

Exercise 5: Suppose you have a class of 12 students. Those students are going to have a race, with 1st place receiving a gold medal, 2nd place a silver medal, and 3rd place a bronze medal. How many possible outcomes for the race are there?
Answer: 12*11*10.

C) sextuples. In all of these variants, the actual options in each stage of the counting process are always the same—meaning, no matter which road you take from A to B, you have the same three modality options, and there will be the same four possible roads from B to C.

Exercise 5 explicitly asks the solver to coordinate students with medals and positions. That is, we can consider a solution to be triples of the form (Gold Student A, Silver Student B, Bronze Student C). In this way, there are 12 possible options for the stage identifying which student receives the (first) gold medal, and, subsequently, 11 options for the second silver medal stage, and finally, ten options for the third bronze medal, leading to 12*11*10 possible outcomes. In contrast to the previous exercises, the options in each stage change depending on the choice(s) in the previous stage(s) (i.e., a student who receives the gold medal cannot also be selected for the silver medal).

Collection 3

Table 3 highlights four nuances (i.e., critical features) of the MP that correspond to the italicized parts of the definition of the MP mentioned previously: *order*, *independence*, *distinct options*, and *distinct composite outcomes*. The range of possible values for each critical feature is also provided. Each critical feature is accentuated by varying that feature, while keeping the other features invariant.

Table 3

Framework for the Critical Features of the Multiplication Principle

Concept	**Critical Feature**	**Range of Permissible Change**
Multiplication Principle for Counting	Order	[We differentiate "outcome vs. process" and "ordinal vs. categorical"] 1. Ordinal outcomes 2. Ordered process and categorical outcomes 3. Unordered outcomes
	Independence	[We differentiate "set vs. cardinality" (i.e., A vs. $\|A\|$)] 1. Independent in set and cardinality 2. Independent in cardinality but not set 3. Not independent in cardinality
	Distinct Options	1. Distinct options 2. Non-distinct options (i.e., at least two options are of the same type)
	Distinct Composite Outcomes	1. Distinct composite outcomes 2. Non-distinct composite outcomes

Order

Although multiplication is commutative, the MP suggests that we adopt an explicit order to the stages in the counting process before we use multiplication (Lockwood et al., 2017; Lockwood & Purdy, 2019). Exercises 6-7 (Table 4) highlight the different ways in which order may appear in counting problems and address the range of permissible change as it relates to order in the MP.

Table 4

Exercises 6-7

Exercise 6: (i) Suppose you are given the letters ARC. How many different arrangements of all three letters are possible? (ii) Now, consider from a class of 12 students, there were 12*11*10 possible outcomes for coming in 1st, 2nd, and 3rd place in a race. For each of these two problems, list 5 outcomes. Then describe what it is that makes one outcome different from another one. *Answers:* (i) 5 outcomes: ARC, ACR, CRA, CAR, RCA. (ii) 5 outcomes: If we label the students A, B, C, …, L, then: ACD, ADC, LAC, KBL, CDE. ***Exercise 7:*** Suppose you have a class of 12 students. (i) The class is now going to select a committee of 3 students to be representatives at a larger gathering. How many possible committees of students from the class are there? (ii) The class is going to select a committee of 3 students to be representatives at a larger gathering. On the committee, there is a President, VP, and Treasurer. How many possible committees of students from the class are there? *Answers:* (i) Not 12*11*10. [Answer is 12*11*10/6.] (ii) 12*11*10.

Exercise 6 introduces arrangements. Critical in parts (i) and (ii) is the arrangement of the outcomes themselves (i.e., each letter in the arrangement has an ordinal position). Order also occurs in general, with each element having an ordinal position. For example, we can compare parts (i) and (ii) on the order dimension. ARC is different from CAR, even with the same letters, because the letters are in a different order. Thus, it is not just the subset of letters included but their different orders that the MP enumerates. Use of the MP links to order and includes, as illustrations, exercises having ordinal outcomes—first position, second position, and so forth.

Exercise 7 considers ranges of permissible change of order in the MP. Like part (ii) of Exercise 6, part (i) of Exercise 7 provides a context of a class of 12 students, but the ordered first, second, and third places from the prior exercise have changed to a 3-person committee in which order is irrelevant. The goal is to help students recognize that 12*11*10 is *not* the correct solution—because the use of the MP would include all possible *orderings*. For example, in part (ii) of Exercise 6, ACD was different from DAC, because in the former, A wins the gold, while in the latter, D wins the gold. In contrast, in part (i) of Exercise 7, we could imagine forming a committee in which ACD is identical to DAC, because they are comprised of the same students. For present purposes, students need not know the correct solution—i.e., (12*11*10)/6. The point is to help students identify order as a critical feature of the use of multiplication in counting.

Part (ii) of Exercise 7 asks students to form a committee of three people whose roles are not inherently ordinal and thus differ from the roles in part (ii) of Exercise 6, which are ordinal. Nonetheless, by using an ordered process in selecting the candidates, first for the Treasurer, then the Vice President, and finally the President, the MP treats the (Treasurer, Vice President, President) triples as though they were ordered. That is, we would interpret any of the 12*11*10 triples as ordered: the first selected student is the Treasurer, and so on. Because we draw from the same set of 12 students, the triples (A, B, C) and (C, B, A) would both be included in the 12*11*10 solutions, and would represent different solutions, since in the former, A is the Treasurer, while in the latter, C is the Treasurer. By varying the type of outcomes, this exercise highlights the way in which ordered processes can coincide with outcomes.

Independence

Independence—i.e., the concept that the number of options at each stage of a counting process must be independent of each other—is a critical feature of the MP. Exercises 8-10 in Table 5 highlight the different ways in which independence may inspire the use of the MP.

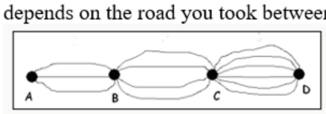
In Exercise 8, the options are independent in both set and cardinality—i.e., to get from B to C, students always choose from the same set of roads, and they always choose from the same number of roads. After listing five outcomes in part (i), students determine in part (ii) whether their choice of road between B and C depends on the road they take between A and B. Using the diagram as a tool, they should recognize that no matter the road they take between A and B, they will *always* have to choose one of the four roads joining B to C—the set of choices they have is independent of any choice of road from A to B. As for part (iii), since the set is the same, its cardinality is too: there are always the same four choices.

Exercise 9 demonstrates a situation that is independent in

Table 5

Exercises 8-10

Exercise 8: Consider the solution to a previous exercise: There were 3*4*6 different routes to get from A to D.
(i) If we label each of the thirteen roads with a unique name (e.g., Abrams Street), give 5 outcomes for routes between A and D.
(ii) Describe whether or not your choice for the road between B and C depends on the road you took between A and B.
(iii) Describe whether or not the number of choices you have for roads between B and C depends on the road you took between A and B.



Answer: (i) Call the streets A, B, C, …, M. Choose from A,B,C first, then D,E,F,G second, then H,I,J,K,L,M last. So: AFM, BEL, BFJ, CDI, CGM. (ii) The choice of D,E,F,G did not depend on the choice for A,B,C. (iii) The number of choices was always 4.

Exercise 9: Each of the following problems has 5*4 as its answer.
(i) Suppose you are given the letters MARCH. Ana is trying to count how many different possible rearrangements of two of the five letters are possible.
(ii) At your local café are 5 types of donuts and 4 flavors of coffee. Suppose you order a breakfast that consists of a donut and a coffee. How many different possible breakfasts could you order?
This means for each of the 5 initial options, there are 4 remaining options. List out the 4 remaining options for each problem. How are those lists different? What do you observe is common between the two problems?
Answers: (i) If M, then A, R, C, H; if A, then M, R, C, H; if R, then M, A, C, H; if C, then M, A, R, H; if H, then M, A, R, C. (ii) C1, C2, C3, C4.

Exercise 10: At your local café are 5 types of donuts and 4 flavors of coffee. Suppose that the café is down to 20 donuts (2 glazed donuts; 4 sprinkled; 1 jelly; 8 cake; and 5 twists), and that you are ordering three donuts, one for Sam, Jenny, and Walter. How many different possible breakfasts for those three friends could you order? Explain why the solution 5*5*5 is incorrect.
Answer: With a limited supply, independence of cardinality does not work (if you choose Glaze, Glaze, then there are no more Glazed donuts and so only 4 options for the third (not 5)). [Actual answer is 111.]

cardinality but not in set. Scenario (b) is analogous to Exercise 1: If the four coffee flavors are {Regular, Vanilla, Hazelnut, Pecan}, then one is always choosing from those same four options, regardless of which donut is selected. Scenario (a), however, is different. If the letter M is first, then the four options for the second letter are {A, R, C, H}, but if the letter A is first, then the four options for the second letter are {M, R, C, H}, and so on. Here, the set of options for the second letter is *dependent* on the choice for the first letter. What is common in scenarios (a) and (b) is that there are always four options—even though they may not be *the same* four options. Thus, independence of the cardinality of the options may come either with the options themselves being independent, as in scenario (b), or with the options changing but always retaining the same quantity, as in scenario (a).

Exercise 10 involves a limited number of each type of donut. Many students may mistake 5*5*5 for the correct solution. The crux of the matter is, however, that the cardinality of options is not independent of prior choices. For example, if you order Sam a glazed and Jenny a glazed, then the café would run out of glazed donuts, and only four donut options would remain for Walter. But if you order Sam a glazed and Jenny a sprinkled, then five donut options would remain for Walter. The number of options changes depending on what happens in prior stages, and we cannot use the MP in this way.

Distinct Options

Distinct options are a third critical feature of the MP. Options are distinct when they are distinguishable at each stage of the process. Exercises 11-12 in Table 6 highlight how distinct and non-distinct options influence the application of the MP.

Exercise 11 highlights distinct and non-distinct options using the letters MASS. In part (i), the proposed solution is not correct because MASS does not contain four distinct letters due to the repetition of the letter S. However, in part (ii), the suggested solution would be appropriate for the letters MAS1S2, because the subscripts provide distinct options. Only when the options at each stage are distinct can we use the MP to count.

In Exercise 12, while the café has 20 donuts, the two glazed donuts are the same, the four sprinkled donuts are the same, and so on. For example, suppose we number the sprinkled donuts D3, D4, D5, D6. Then the 20*19*18 solutions would include (D3, D4, D5), (D5, D4, D3), and so forth, which are really all *one* solution: Sam, Jenny, and Walter each get a sprinkled donut. By using a familiar context and keeping all other critical features the same, one can distinguish a situation that contains non-distinct options from a situation that contains distinct options.

Distinct Composite Outcomes

The final critical feature of the MP is distinct composite outcomes. Here, we consider the distinctness not just of the options at each stage in the process, but also of the composite outcomes resulting from completing all stages of the process. From a set-theoretic perspective, this represents whether a one-to-one correspondence exists between the counting process and a set of outcomes representing the solution. See Table 7.

Exercise 13 illustrates the notion of distinct composite outcomes. Even though the donut options are the same for each person, when we consider the composite outcomes, they are distinct. A three-stage process would be to pick a donut for Sam, then Jenny, then Walter, resulting in (Sam's donut,

Table 6

Exercises 11-12

Exercise 11: Determine the number of different rearrangements of the four letters of MASS. (i) Explain whether the solution 4*3*2*1 is correct or not. (ii) If we differentiated the S's as S1 and S2, would 4*3*2*1 count the number of rearrangements of the letters MAS_1S_2? *Answer:* (i) Not correct. There are not 4 "distinct" options. (ii) Yes, correct. There are now 4 "distinct" options.
Exercise 12: At your local café are 5 types of donuts and 4 flavors of coffee. Suppose they are down to 20 donuts (2 glazed donuts, 4 sprinkled, 1 jelly, 8 cake, and 5 twists), and that you are ordering three donuts, one for Sam, Jenny, and Walter. How many different possible breakfasts for those three friends could you order? Explain why the solution 20*19*18 is incorrect. *Answer:* 20*19*18 is not correct because the options are not distinct—there are not 20 different donuts—the 2 "glazed" donuts are the same. [Actual answer is 111.]

Table 7

Exercises 13-14

Exercise 13: At your local café are 5 types of donuts and 4 flavors of coffee. You are ordering three donuts, one for Sam, one for Jenny, and one for Walter. How many different possible breakfasts for these three people could you order? *Answer*: 5*5*5.
Exercise 14: Suppose you have a class of 12 students, 8 of whom are American, 2 European, and 2 Asian. The class is going to select a committee of 3 students to be representatives at a larger gathering, with at least 1 Asian and 1 European student on the committee. How many possible committees of students from the class are there? *Answer*: 1*2 + 1*2 + 2*2*12.

Jenny's donut, Walter's donut) triples. The solution 5*5*5 gives five distinct options for each of the three stages (i.e., the three people). So, a composite outcome might be (glazed, sprinkled, glazed). This is different from another of the composite outcomes (sprinkled, glazed, glazed) because Sam is getting the sprinkled donut instead of Jenny. Thus, the composite outcomes are all distinct from each other.

In Exercise 14, composite outcomes are not all distinct. A seemingly reasonable process would be first to select one of the two Asians to be on the committee, then select one of the two Europeans, and then select one of the ten remaining students (from one Asian, one European, and eight Americans). This process produces a solution of 2*2*10 = 2*2*(1+1+8), which results in overcounting the outcomes because a one-to-one correspondence does not exist between the counting process and the number of outcomes. That is, the composite outcomes are not all distinct. For example, this process would yield both (Asian 1, European 1, Asian 2) and (Asian 2, European 1, Asian 1). However, these composite outcomes are the *same* committee. When the ordered process produces the same solution in more than one way, we cannot use the MP to enumerate the solutions because the composite outcomes would not all be distinct. That is not to say the exercise is unsolvable. Breaking the exercise into three distinct cases, we can use the MP in each of those three cases to obtain a correct count. There are 1*2 committees with two Asian students and 1 European student, 1*2 committees with two European students and 1 Asian student, and 2*2*8 committees with one student of each nationality.

Collection 4

The last collection of exercises highlights the interrelatedness of the four critical features. Exercises 15-17 in Table 8 illustrate how those critical features work together to build a deeper understanding of the MP for counting. We comment on Exercise 15 but leave Exercises 16 and 17 for the reader to contemplate.

The three parts in Exercise 15 draw out three critical features of the MP: independence, order, and distinct composite outcomes. The solution in part (i), 5*4*3 presupposes that you cannot choose two of the same type of donut, and hence that the number of available options at later stages is dependent on the number of available options at earlier stages. The exercise, however, explicitly allows you to order as many of one type of donut as you like. The solutions in parts (i) and (ii) presuppose order and distinct composite outcomes. For example, the 5*5*5 solution in part (ii), which may tempt some students, would generate (glazed, glazed, sprinkled) and (glazed, sprinkled, glazed) as distinct composite outcomes because their order is different. However, they are *not* different breakfasts. Similarly, the 5*4*3 solution in part (i) would generate, for example, (glazed, sprinkled, jelly) and (glazed, jelly, sprinkled) as distinct composite outcomes when, in fact, they are the same breakfast. Thus, the critical features of order and distinct composite outcomes in the MP are sometimes linked. The correct solution in part (iii) requires dividing the exercise into three cases, based on independence—when we have a) three donuts of the same type, b) two donuts of the same type, and c) three different types of donuts. If we have three donuts of the same type, then once we select the first donut—in one of five ways—there is only one option for the remaining donuts. If we have two donuts of the same type, then once we select those two donuts—also in one of five ways—there are four options for the remaining donut. After multiplying 5*4*3 for the third option, division by 6 is needed to ensure that only one of the six possible orderings is being counted. Thus, the cases themselves are constructed to help navigate the issue of the number of outcomes being independent of previous choices so that the MP can be used within each of the three cases.

Table 8

Exercises 15-17

Exercise 15: At your local café are 5 types of donuts and 4 flavors of coffee. You decide to order a breakfast consisting of three donuts, and you can order as many of one type of donut as you like. How many different possible breakfasts could you order? (i) Explain why 5*4*3 is an incorrect answer. (ii) Explain why 5*5*5 is an incorrect answer. (iii) Find the correct answer. *Answer*: (iii) (5*4*3)/6 + 5*4 + 5.
Exercise 16: Suppose you have a class of 12 students. The class is going to select a committee consisting of some number of students to be representatives at a larger gathering. The committee could have 0, 1, 2, 3, … up to 12 students on it. How many possible committees of students from the class are there? *Answer*: 2*2*2*...*2 = 2^12.
Exercise 17: Suppose you are given the letters AEIOUJKLMN. (i) How many different arrangements of 4 of the ten letters are possible? (ii) How many different arrangements of 4 of the ten letters are there that do not begin with A and that end with a vowel? *Answer*: (i) 10*9*8*7. (ii) 8*8*7*4 + 9*8*7*1.

Discussion and Conclusion

This problem-based curriculum was designed as an instructional sequence to develop the MP. More broadly, this sequence of exercises is aligned with variation theory (e.g., Marton, Runesson, & Tsui, 2004). Variation theory describes learning as a process of considering variation against a backdrop of invariance. Specifically, variation theory has four stages: (1) contrast, (2) generalization, (3) separation, and (4) fusion (Marton, Runesson, & Tsui, 2004). The theory suggests that learning a new concept, such as the MP, should begin with contrast, move to generalization, on to separation, and lastly, fusion. The collection of problems presented in this paper aligns with these four stages (i.e., Collection 1 aligns with contrast, Collection 2 aligns with generalization across different contexts, Collection 3 aligns with separation when critical features vary, and Collection 4 fuses those critical features).

The exercises are engineered for a PBL setting. Instructors would assume the role of facilitator as students work independently and collaboratively to construct their own knowledge rather than listening to a traditional lecture (Torp & Sage, 2002). Furthermore, the exercises are meant to set up an explicit, full-class discussion through student presentations and discussion of solutions. This approach will collectively surface the conceptual conclusions about the MP—e.g., using students' solutions to the exercises to elicit the critical features, their subtleties, and their importance for recognizing when and how to apply the MP while solving counting exercises.

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