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Considering Opportunities for Mathematical *Magic:* Design Principles for a Mathematician-Early Career Teacher Mentoring Model

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ABSTRACT Mathematical *magic*, or experiences of beauty and creativity for K-12 mathematics students, can contribute to deepened content understanding. However, creating opportunities for mathematical *magic* in mathematics classrooms has been a challenge for classroom practitioners. To support development of these experiences in classrooms, we deployed a bi-directional support program between a college-level mathematician and an early-career secondary teacher. Based on data collected from this project, we have identified four principles that will promote mathematical *magic*. We conclude that mathematicians and early career teachers can work together to a) leverage understandings of the nature of mathematics, b) apply meta-mathematical reasoning, c) share knowledge of content and students across contexts, and d) recognize students' mathematical development to design learning experiences for mathematical *magic*.

KEYWORDS: *Mentoring, early-career teachers, mathematics faculty*

[W]hat's really challenging for me is one of the best things that I did for my own education was these number theory classes and these proof writing classes. I feel a lot of the *magic* of it is missing from my classes. You know, the power of looking through data and looking through numbers and coming up with some conjecture, then proving or disproving it. And that intense feeling, although it has been discovered before, in the moment, you feel like you are a genius. Or you are the only one that's ever discovered it. It's almost like a feeling of creation [...] It's how I imagine maybe artists feel sometimes, when they feel like they've made something through their own devices.

– Ms. Jackson, a teacher participant

Experiences of mathematical *magic*, or the creativity and beauty in mathematics, as described by Ms. Jackson, have been associated with positive mathematical identities (Palmer, 2009), increased interest in mathematics (Brinkmann, 2009), and deepened mathematical understanding (Cellucci, 2015). These approaches align with how mathematicians describe their own disciplinary experiences: beautiful, personal processes of doing (Burton, 2007). However, this kind of *magic* is often missing from secondary classrooms, according to teachers (Karp, 2008). At the post-secondary level, some mathematicians describe teaching as procedural communication where students must identify the underlying beauty (Burton, 2007). Therefore, students at the K-16 level do not always experience mathematical *magic*, despite its importance to both teachers and mathematicians. Further, a majority of the research about college-faculty mathematicians in mathematics education has been in their own post-secondary contexts (e.g., Burton, 2001; Nardi, 2008; Sfard, 1998), rather than at the K-12 level, so how they might support beginning teachers is not widely understood. Here, we propose to examine how the dual expertise of a beginning teacher and a professional mathematician in a mentoring relationship might be leveraged to produce mathematical *magic* within secondary schools.

To better understand how such an interaction might look, we completed an exploratory study associated with Exemplary Mathematics Teachers for High Needs Schools (EMTHNS). As part of a larger project, teachers in their first four years of teaching were mentored by both college-faculty mathematicians and experienced secondary mathematics teachers. In this paper, we focus on the teacher-mathematician relationship and propose four principles that leverage the expertise of mathematician and teacher to create a vision for mathematics classroom practice that includes mathematical *magic*. In what follows, we will begin with the relevant literature and describe the complete program in which teachers and mathematicians participated. Based on data analysis, we propose design principles for similar mentoring programs. The concluding section presents implications for the field.

Situating the Challenge of Mathematical *Magic*

This project is situated in the literature about experiences of the aesthetic of disciplinary mathematics and mentoring of early career mathematics teachers. Here we explain how the concept of mathematical *magic* relates to the literature on aesthetics in K-12 classrooms. We will also describe some of what is known about faculty mathematicians in mathematics education research and mentoring early career mathematics teachers.

Considerations of aesthetics in mathematics classrooms

The concept of mathematical *magic*, the appreciation of the aesthetic of mathematics, is important for teachers because it allows them to see themselves in the discipline (Hobbs, 2012). For learners, aesthetic considerations have been shown to support the development of a positive mathematician identity (Palmer, 2009) and to stimulate interest in mathematics (Brinkmann, 2009). Mathematical aesthetic also can provide a route toward a deeper understanding of the discipline (Cellucci, 2015). Additionally, it is essential to mathematical creativity (Brinkmann & Sriraman, 2009) and the judgement of problem posing in mathematics (Crespo & Sinclair, 2008). For professional mathematicians, mathematics is a beautiful, personal process of doing or, in our words, *magic* (Burton, 2007).

At the post-secondary level, for some faculty, mathematics teaching is procedural communication where students leverage procedure to identify underlying beauty (Burton, 2007). Other research centers teacher candidates' development of content and pedagogical knowledge through problem solving and inquiry (Apkarian et al., 2023), developing mathematically-focused art (An et al., 2023), or conjecture-building activities (Meagher et al., 2020) that could lead to mathematics *magic*. Thus, PTs may or may not have been exposed to experiences of mathematical *magic*.

Similarly at the K-12 level, mathematical aesthetics have not always been a central concern (Sinclair & Crespo, 2006). In fact, Karp (2008) argues that teachers suggest beautiful mathematics exist outside of the mathematics curriculum. However, recent studies attempt to better define creativity and determine how to encourage it at the K-12 level (e.g., Assmus & Fritzlar, 2022; Lu & Kaiser, 2021). Furthermore, practices that support *magic* are not sufficient to provide equitable and empowering mathematics experiences for traditionally underserved students because attention to students' cultural backgrounds is critical (Battey, 2013). Therefore, there is space for collaboration to ensure equitable approaches to creativity in the K-12 classroom.

Faculty mathematicians in mathematics education research

Mathematicians target two major ideas in courses for prospective teachers (PTs) to understand: the mathematical content and the meta-mathematical experiences, such as the aesthetics, proof, and uses of mathematical definitions (Leikin et al., 2018). However, a lecture does not necessarily convey these kinds of messages to students, even at the post-secondary level (Lew et al., 2016).

In parallel research, due to their different professional roles, professional mathematicians tend to attend to *what* students learn, while teachers tend to focus on *how* students learn (Wade et al. 2016). Therefore, mathematicians, mathematics educators, and mathematics researchers can use different expertise to support learners (Bass, 1997). Mathematicians involved in mathematics education act at the boundaries to transform classroom practice, in both K-12 and the post-secondary classrooms (Darragh, 2022). This mentoring program seeks to support participating teachers and mathematicians to work synergistically to support mathematical *magic*.

Existing mentoring models

Mentoring and induction programs have supported early-career teachers to stay in the profession, increase scores on classroom practice measures, and improve student achievement (Ingersoll & Strong, 2011). Other benefits for both mentor and mentee are productive relationships with colleagues, reflection on practice or subject matter, professional development, and personal satisfaction (Ehrich et al., 2004). Additionally, field-based mentoring programs can encourage new teachers to change beliefs about groups of students (Garza & Harter, 2016). A constructivist mentoring relationship, rather than transmission-style, supports first-year teachers with their self-efficacy, enthusiasm, job satisfaction, and emotional support (Richter et al., 2013). Reported challenges for mentees and mentors include lack of time, professional or personal mismatch, a non-productive relationship, and additional programmatic commitments (Ehrich et al., 2004).

Research suggests that a content mentor plays a unique role in developing content knowledge and could bolster novice teachers in developing student disciplinary thinking (Achinstein & Davis, 2014; Wang & Paine, 2001). There is limited research on mentoring between professional mathematicians and beginning teachers. In reporting on Math Circles where faculty and teachers worked together, college faculty described extending their pedagogical knowledge and conversations and teachers identified deepened content knowledge as well as expanded repertoires of classroom approaches (Cushman et al., 2023). Further, a mentoring model between PTs and faculty has suggested, among other parameters, that well-established mentoring guidelines and clear communication channels between stakeholders in the project strengthen the program (Hodge et al., 2019). In aligned research on apprenticeship programs that invite science teachers to spend summers assisting scientists an increase in discourse practices, collaboration, changes in classroom practices (Sadler et al., 2010), changes in pedagogical beliefs (Miranda & Damico, 2013), and beliefs about the nature of science (Baker & Keller, 2010) have been identified. In sum, teacher exposure to subject matter experts can positively impact their classroom practice, when appropriately framed. The program described here was designed to benefit to both the teacher and the mathematician as described by Cushman et al. (2023).

Initial mentoring model and purpose

The mentoring model described here was part of a larger preparation and professional learning program designed to support beginning teachers. The five-year program was created to support individuals with strong content preparation to enter and stay in secondary teaching. The two-way mentoring model was conceived to provide beginning teachers with both a mathematician and a master teacher mentor, starting from their preservice year and continuing during their first four years of teaching (Albert, 2019). In the first year of the program, beginning teachers completed a one-year Master's program and associated student teaching for an initial state credential. Each fellow joined a "high needs" district, as defined by grant funders, in a small or medium sized city. During subsequent years, in addition to mentoring, beginning teachers participated in a professional community of mathematicians and mathematics educators attending monthly meetings and four research colloquia per year. In the monthly seminars, mentors and mentees examined problems of practice in the context of contemporary research. The mathematics education colloquia featured practitioners, educators, and researchers in the field of mathematics or mathematics education. The focus of this study is on the mentoring relationship between the beginning teacher and the mathematician.

For the faculty-teacher mentoring pair, pairs met at least twice per year for a pre-meeting, a classroom observation, and a follow-up meeting. Mentors and mentees were also able to contact each other virtually. Mathematicians were not required to use an observation protocol, but were provided with questions to direct their observations. Additionally, pairs attended monthly seminars and colloquia together.

Methods

The study used qualitative, inductive content approach, guided by Elo and Kyngäs (2008), to extract design principles for a mentoring program between a beginning teacher and faculty mathematician that promoted mathematical *magic*. The data comprised an approximately 60-minute, individual, semi-structured interview with each participant. Participants included seven beginning teachers and three mathematicians. Each teacher had an extensive mathematics content background – six held undergraduate mathematics degrees, and one an engineering degree. Multiple fellows had double majors in subjects including English literature and art. However, the teachers' experiences in schools were more varied, with some who had previously been classroom teachers and others who had not worked in schools professionally. At the time of their interviews, faculty mentors had 10 to 40 years of university experience. Two were current faculty, and one was retired. Two were primarily research mathematicians, while one was primarily a teaching professor.

Because this was a new program, we took an exploratory approach to analyzing the data, using an inductive content analysis, starting with open coding, creating categories and associated memos, and abstracting themes, in this case design principles, from interviews (Elo & Kyngäs, 2008). Further, because the purpose of these analyses was to improve the mentoring between a mathematician and a teacher, special attention was paid to the interactions around mathematics content and pedagogy. Known issues in mentoring were not targeted because while significant, we focused on the unique aspects of this program.

Results

From the analysis, it was apparent that both participating teachers and mathematicians wanted to share with their respective students' ideas of *mathematical magic*. Participants shared a common love for the discipline of mathematics and found mathematics intellectually satisfying. The focus of the results section are the four emergent principles for designing a similar program.

Principle 1: Determine what constitutes mathematical *magic*

All participants described complementary ideas of what constitutes mathematical *magic*. One participating mathematician suggested that mathematical *magic* emerges by using what is known to understand what is unknown. He stated, "mathematics is a lot about pushing boundaries. [...] What happens if we push our boundaries and look for larger and larger structure or different sort of universes to explore?" The mathematician emphasized both a disciplinary principle, examining mathematical structure, and a creative process, problem posing.

A teacher described a complementary conception mathematical *magic* – leveraging patterns to identify generalizations. The teacher stated, "finding ways to capture patterns that we see […] finding ways to create structure out of the abstract ideas." This teacher also provided a disciplinary practice, pattern seeking, and a creative process central to the discipline, generalization.

In undertaking a mentoring program, we suggest that teachers and mathematicians discuss what mathematics is and their own experiences of mathematical *magic*. The focus should be on both the central disciplinary concerns and how those concerns can support creative mathematical endeavors. As they come to a shared definition of mathematical *magic*, mentoring pairs can then proceed to think about lesson designs that might support it.

Principle 2: Apply knowledge of metamathematical reasoning to design for mathematical *magic*

Both participating teachers and mathematicians believed that the teachers' mathematical subject matter knowledge was well developed for their careers as secondary teachers. However, all participants thought mathematicians could bring expertise in metamathematical reasoning to the K-12 classroom. Metamathematical reasoning is an understanding of the organization of the many sub-fields of mathematics, how mathematical knowledge relatively situated, what should be focused on, and how to differentiate between conventions, axioms, theorems, and so on (Dawkins & Roh, 2016). Mentoring pairs felt this metamathematical expertise would be useful to support beginning teachers to a) locate mathematics content in the mathematical universe and b) identify the kinds of skills, habits, procedures, and understandings that would best serve secondary students.

One teacher described that completing a more rigorous version of a high school proof would help her to, "get some better intuition looking at these high school concepts and how can I connect them to something that I might not be thinking of." Thus, mentoring could help beginning teachers understand how to fit a particular mathematical idea into a larger mathematical world, supporting their continuing development of metamathematical reasoning.

Secondly, while teachers bring expertise in the design of their secondary courses, a mathematician might aid a teacher to understand how to organize and locate those concepts within the mathematical universe beyond the existing curriculum. One mathematician thought he could do this be encouraging task design that supports students to ask questions like, "[Is this] something that works in a specific case or works in a specific type of situation, but does it work in a broader class of situations? How?" In this way, the mathematician and teacher could collaborate to identify how the mathematical topic might have implications for additional study.

Finally, metamathematical reasoning is also useful to identify mathematical objects, procedures, or habits that would serve secondary students to achieve a deep level of mathematical understanding. A beginning teacher described the metamathematical guidance that she would hope to gain in the mentoring relationship,

> What are...essential things that you want to instill [in the secondary classrooms] so that later on, you will

be successful? How would you really think about that in a purely mathematical way, including those mathematical concepts that students need to know or maybe we know how to teach these concepts better?

Specifically, the teacher was seeking mathematically-focused advice around which aspects of an idea should be emphasized and extended.

A focus on metamathematical reasoning could provide additional support for mathematical *magic*. Specifically, a deeper understanding of the mathematical universe might provide secondary students with additional skills to recognize the creative possibilities and intellectual challenges in mathematics. For the mentor pairs, we would suggest programming to discuss and identify metamathematical reasoning.

Principle 3: Sharing knowledge of content and teaching/students across contexts to create opportunities for mathematical *magic*

The mentoring pair described that there were some shared practices that translated between secondary and post-secondary contexts. One mathematician described it as bringing, "the experience of a million years as a math teacher." Based on the data, a mathematician's teaching experience might provide transferable expertise in a) describing and organizing problem-solving approaches, b) identifying and addressing student misconceptions, and c) sequencing and tying content together. According to one mathematician, "[a]n advantage for me is the sense that they know I can't really comment with any depth of knowledge about managing the classroom" or other aspects of the secondary context, but can really focus on the transferable expertise.

However, transferring between contexts can be challenging. One mentoring pair had a conversation connecting curricular content to undergraduate mathematics, but the beginning teacher left the conversation thinking, "I would love to share the information that you're telling me, but how could we break it down [for secondary students]?" On the other hand, another beginning teacher was struggling to help ninth-grade students understand formulas for surface area and volume without using ideas from calculus. After a conversation, the beginning teacher reported that the mathematician-mentor, "was definitely helpful in giving me ways that I can answer those questions to a group of students without the use of calculus." While many mathematics teachers might provide this kind of support, the faculty mathematician's focus on mathematical ideas, rather than the secondary context, provided an opening for this conversation.

To address the challenges of translating expertise across various mathematical contexts, we suggest creating a space for explicit conversations about which student experiences of content are parallel, which are transferable with consideration, and which are not. Through these conversations, mentor pairs can begin to address how to design lessons that best harness their joint expertise and provide fertile ground for mathematical *magic*. Sinclair (2003) suggests that providing diverse perspectives on particular mathematical concepts, these discussions may also encourage real mathematical learning in addition to maintaining teachers' and students' interest in the content. In future iterations, beginning teachers might visit their mathematician-mentors' classrooms to initiate these conversations. Teachers and mathematicians can work alongside each other to design classroom experiences that highlight the *why*, which may lead to innovative experiences for themselves and students across contexts.

Principle 4: Embracing teachers' expertise in students' mathematical development to create mathematical *magic*

Throughout the mentoring experience, mathematicians repeatedly noted that the beginning teachers had expertise in supporting their students and managing the numerous other requirements for teachers. One mathematician noted that in the face of their mentee's classroom challenges, "it makes me kind of question my mantra that the math content is always primary." In addition to the pedagogical expertise, mentors also referenced how teachers' knowledge of adolescent development influences their content decisions. The difference between secondary and post-secondary levels was described as "a different tenor," by one mathematician, saying, "it is also part of connecting to mathematics as a whole. […] I like that it is a vertical viewpoint." Beginning teachers need to understand both their students' psychological development and mathematical learning trajectory.

Given what teachers know about students' development trajectory, a mathematician recognized that this was displayed in how teachers choose to deploy mathematical activities. In the following, a mathematician outlines how the teacher he was mentoring used her knowledge of students to prepare them to learn. The mathematician stated:

> [The students] played an online game and they had to name the type of triangle […] And I thought, you know, this teacher's done no content and just teaches

> them terminology. And then I noticed that the students really liked the game, and they really liked that they could name the different types of triangles. [...] I was very surprised at how successful that was, and the last, maybe a third of the class, she's starting to talk about theorems in geometry [...] and the students were ready to listen.

Because university students are primarily adults, it is assumed they will be primed to learn the material as it is being presented. For this mentee, though, it was important for the students to be successful before they were ready to learn. Another mathematician expressed a similar sentiment, pointing out that his mentees needed to "intentionally [engage] individual students by asking questions or by talking to them," which was different from his experience at the university level.

We anticipate that the application of this design principle will support Principle 3. Beginning teachers have expertise in how individual development and engagement mediates the learning of the content. When considering how to design for mathematical *magic*, teachers' knowledge of their students and the resulting developmentally appropriate a sequence are essential. And while mathematical *magic* is a worthy goal, it might be experienced in very different ways by different students, and thus, teachers' expertise will be invaluable in any curricular designs.

Mentoring Program Challenges

In this section, we will highlight the two major challenges that arose in the program. These challenges are separate from the design principles, but still focused on the context of a mathematician-beginning teacher pair. Beginning teachers and mathematicians, by program design, work in different institutions. This results in two challenges that include: a) not being in proximity when challenges arise and b) having allied but not identical professional roles.

The first challenge arose when beginning teachers noted issues, in planning or classrooms, that need to be addressed quickly and cannot always be predicted. The mathematician mentors were not necessarily in the teachers' immediate physical space when such questions arose, due to working at different institutions. Some of this can be addressed by scheduling regular one-on-one meetings and encouraging video conferencing or other virtual mode of communication. We anticipate that this will increase the accessibility and regularity with which teachers can approach mathematicians for support.

The second, perhaps more challenging, issue was that some teachers felt that the mathematicians could not necessarily engage with issues that they faced as beginning teachers. For example, mathematicians did not necessarily have experience with "students specifically two, three, four grade levels behind" their peers. Secondary teaching is done within the context of schools with adolescents, while mathematicians do mathematics in research groups or teach adults. Generally, the mathematicians had little or no exposure to schools, beyond their own or their children's schooling. While each of them had an investment in mathematics education, most of their professional time was not spent in K-12 schools. Both beginning teachers and mathematicians noted that the mentoring experience providing a grounding in the realities of classrooms for mathematicians. However, we suggest that in a revision of this program, we can unpack these different professional experiences and center the conversations in mathematical *magic*.

Limitations of the Current Study

Because our teachers serve populations of students that included racial, linguistic, and SES diversity in districts identified as "high needs," attention to issues of power and equity in mathematics classrooms is required for their work and also need to be addressed in undergraduate and graduate education. During the course of the larger mentoring program, mentoring pairs attended colloquia focused on equitable practices in mathematics teaching. Despite this, the mentoring pairs did not report an explicit focus on equitable practices in the teaching of mathematics. The lack of attention to equity and justice issues in conversations around mentoring may have contributed to the teachers' ideas that mathematicians could not interact about the realities of their schools.

In our full data set, issues of structural inequality and effects on the classroom were mentioned briefly in the mathematicians' observations. This discussion did not extend to help learners navigate their realities or how the various identities of these learners might impact their experiences in mathematics. We suggest continuing to provide additional programmatic reinforcement to support growth in both sets of professionals. Considering equity is an essential part of designing for mathematical *magic*.

Closing Thoughts

The principles outlined here are not the only way that a mathematician and a secondary teacher can contribute to a mentoring relationship through mathematical *magic*. Some of our lingering questions relate to issues that arise from this relationship, and these questions may or may not be related to this goal. However, this kind of mentoring relationship can be useful because it provides some direction for doing something that both teachers and mathematicians, at least those who are participating, wanted to do but were unsure of how to complete.

Additionally, we do believe in the promise of this goal. While the aesthetics of mathematics have previously been identified as frivolous or elitist, they are, in fact, closely linked to areas of student engagement and intrinsic motivation that, when carefully considered, could have a social justice purpose in mathematics education (Sinclair, 2009). Lockhart (2009) argues that students are often required to engage in formalism throughout their K-12 schooling before they are given the opportunities to experience mathematical creativity in post-secondary education. Lockhart has been criticized for not focusing on addressing these issues systematically, rather than individually (Tossavainen, 2014). For these mentoring pairs and a larger program, mathematical *magic* could be supported in the 6-16 academic space, providing a model for a systemic shift.

Additionally, K-12 students have limited conceptions of the work of a mathematician (Latterell & Wilson, 2012). When students do have an opportunity to interact with mathematicians, they came to understand mathematicians' work and expressed more interest in doing this work in their future (Latterell & Wilson, 2012). However, in a previous study with the same participants, we discovered that the mentorship pairing of mathematicians and beginning teachers was a power-free, two-way mutually beneficial connection that was bound by their love of mathematics (Albert, 2019). This aspect embraces the aesthetic nature of mathematical *magic*. In the future, we suggest that mathematicians are given some opportunities to discuss what they learned, regarding the experience of secondary mathematics classrooms, from their visits with their mentees. In addition, beginning teachers could provide opportunities for students to occasionally interact with mathematicians during their classroom visits. This would provide an additional strength for this project.

Finally, and perhaps most importantly, when experiences in designing courses that can support students to experience mathematical creation are combined with engaging students and understanding their contexts, mathematical *magic* can be born. Leveraging the expertise of both teachers and mathematicians, tasks can be designed and implemented with a focus on the aesthetic. Providing this explicit focus will help mathematicians and beginning teachers think through these experiences and their representation at the secondary or early post-secondary mathematical level. These interactions will also allow us to create a reproducible model of designing and implementing these kinds of explorations that embody mathematical *magic*.

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