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Nikolai Bugaev's Philosophy of Education: *Arithmetic of Whole Numbers* Textbook Analysis

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ABSTRACT This article explores the educational and philosophical contributions of Nikolai V. Bugaev, a prominent 19th-century Russian mathematician and founder of the Moscow philosophical-mathematical school. The study specifically focuses on Bugaev's textbook, Arithmetic of Whole Numbers, analyzing Bugaev's pedagogical approaches within the broader context of Russia's educational reforms during that era. Bugaev's work can be seen as a response to the evolving needs of a rapidly industrializing society, in which he emphasizes three fundamental components of mathematical education: integration of theory, calculation mechanisms, and practical problem-solving. While Bugaev's textbook may not have achieved the widespread popularity of other contemporaneous works, it played a crucial role in fostering mathematical thinking and underscored his vision of mathematics as a tool for intellectual development and its interconnectedness with other fields of knowledge.

KEYWORDS: *Russian mathematics education, textbook analysis, philosophy and pedagogy*

Nikolai V. Bugaev (1837-1903) was a distinguished Russian mathematician, chairman of the Moscow Mathematical Society, and founder of the Moscow philosophical-mathematical school. His life's work coincided with the second half of the 19th century, a period of significant social, political, and economic changes in Russia, all of which influenced the rapid development of the country's education system. Bugaev significantly contributed to the field of education, specifically mathematics education, giving numerous speeches about the Russian education system and writing a series of school mathematics textbooks, from arithmetic to geometry. Although Bugaev is recognized as an outstanding mathematician, his name is rarely mentioned in the context of school mathematics education. This article aims to contribute to a better understanding of mathematics education, specifically arithmetic, in Russia during this period, and to examine Bugaev's pedagogical and philosophical approaches to school mathematical education through the example of his textbook, *Arithmetic of Whole Numbers*. In what follows, the author will refer to original documents written by Bugaev, or studies about the mathematician and his textbooks.

Late 19th Century: Development of School Education in Russia

In the 19th century, Russia saw four major reforms in its educational system, drastically influencing mathematics education. After the first two statutes of 1804 and 1828, quality school education (i.e., gymnasia) was limited to a select group of individuals, typically those from families with a certain level of economic and social standing, though not necessarily a very high level. Mass education in Russia began with the third reform, the 1864 Statute on Secondary School, initiated under Emperor Alexander II, following the 1861 abolition of serfdom in Russia. Restrictions on admission to classical gymnasia were mainly lifted, extending to a larger circle of students, and establishing various forms of gymnasia. One of the types was classical gymnasia, with a mission of learning ancient languages and mathematics to promote the growth of intellectual abilities, while the other type was real schools (from the German *Realschulen*), focusing on technical and natural sciences rather than classical languages (Karp, 2013). During this reform period, mathematics played an important role in both types of gymnasia, as well as in other institutions such as commercial, eparchial, and military schools. In classical gymnasia, mathematics was regarded as a formal subject that promoted cognitive growth and remained unaffected by immediate political trends, whereas in real schools, it was essential for aspiring technical specialists and natural scientists (Karp, 2013).

Throughout the 1860s, the mathematics curriculum addressed different objectives depending on the type of school system and the varying needs of each system's graduates. The divergence in the goals of mathematics education became even more pronounced in the early 1870s with the issuance of new regulations by Dmitry Tolstoy in 1871 and 1872. The 1871 statute and 1872 programs under Dmitry Tolstoy diminished the privileges of real school graduates in favor of gymnasium graduates and reduced lower-class student representation from 53% to 44% over the next decade (Karp, 2013). Nevertheless, by the end of the 19th century, gymnasia remained Russia's most prevalent school system. Despite the slow expansion of the mass education system, the overall socio-economic development and industrialization of the country spurred significant growth in new mathematical education, methodologies, programs, and teaching materials for the emerging mass Russian school system. Developing new courses in arithmetic was central to the expansion of mass education, as arithmetic was seen as the foundation of cognitive development.

This development attracted the attention of professionals from various fields, from teachers to university mathematicians. These included diverse individuals such as secondary education teachers A.P. Kiselev, F.I. Simashko, and A.F. Malinin, as well as university professors like A.Y. Davydov, V.A. Yevtushevsky, and N.V. Bugaev (Karp, 2012). It was precisely in this environment that Bugaev's textbook, *Arithmetic of Whole Numbers*, was published in 1875. His approach not only addressed the educational demands of the time, but also anticipated modern pedagogical trends that emphasize critical thinking and problem solving–ideas that align with his broader views on mathematics education, which are explored later in this paper.

Bugaev's Life and Career

Nikolai Vasilyevich Bugaev was born on September 14, 1837, in Dusheti, located in the Tiflis Governorate, an area that is now part of modern-day Georgia. Bugaev was the son of a military doctor, and the father of famous Russian writer, Andrey Bely. Bugaev's educational journey began at the prestigious First Moscow Gymnasium, where he graduated with a gold medal in 1855. He then entered Moscow University, completing his degree in 1859 in the Department of Physics and Mathematics, where Bugaev studied under the guidance of leading Russian professors and educators, N.E. Zernov, N.D. Brashman, and A.Y. Davydov (Kolyagin & Savina, 2009). These scholars organized the Moscow Mathematical Society in 1867, intending to promote the development of mathematical sciences in Russia. From 1859 to 1861, Bugaev studied at the Nikolaevskaya Engineering Academy, seeking to gain practical experience in how mathematics is applied to military engineering, where he attended lectures by the renowned Russian mathematician, M.V. Ostrogradsky (Kolyagin & Savina, 2009).

Following the defense of his master's thesis at Moscow University in 1863, Bugaev spent from 1863 to 1865 in Germany and France, preparing his doctoral dissertation. During this period, Bugaev attended lectures by some of the best European mathematicians of the time, such as Betrand, Kronecker, Kummer, Liouville, Weierstrass, and others, greatly influencing his scientific interests (Kolyagin & Savina, 2009). Initially, Bugaev was quite fascinated with mathematical analysis. After his trip to France, Bugaev's mathematical interests shifted to number theory, and his doctoral dissertation, which he defended in 1866, was devoted to number theory (Kolyagin & Savina, 2009). From 1867 onward, Bugaev worked as a professor at Moscow University, serving twice as dean of the Department of Physics and Mathematics. In 1891, he was elected president of the Moscow Mathematical Society, and in 1897, he was elected a corresponding member of the St. Petersburg Academy of Sciences (Kolyagin & Savina, 2009).

Bugaev developed a whole new perspective of studying functions by creating a new approach, which he initially named the 'theory of discontinuous functions' and later renamed 'arithmology.' According to Bugaev, arithmology represents the discontinuity of the surrounding world and allows for the application of mathematics in understanding all fields of knowledge (Taube, 1907). This paper does not seek to offer a deep analysis of Bugaev's philosophical concepts; rather, it focuses on how these ideas shaped his overall approach to mathematics education, with particular emphasis on their influence in his textbook, *Arithmetic of Whole Numbers*.

Bugaev's Views on Mathematics Education

Bugaev's pedagogical involvement began immediately after his return from his time abroad in 1865. His new position as an associate professor at Moscow University required him to deliver an inaugural lecture on number theory. At the beginning of his speech, Bugaev emphasized the importance of defining any scientific theory from a historical perspective of the theory's development and understanding its role within the overall system of mathematical sciences (Bugaev, 1877). Although Bugaev expressed these ideas in relation to number theory, they reflected his general philosophical stance on the process of mathematical cognition.

Bugayev's philosophical views changed throughout his life–in the 1860s, he was still a follower of classical positivism, a theory by French philosopher, Auguste Comte, asserting that true knowledge can only be derived from empirical evidence obtained through scientific methods and dismissing metaphysical and theological approaches as ineffective. Yet, in subsequent years, Bugaev distinguished his understanding from the positivist view:

> The so-called positivist worldview seeks to answer only the question: 'How do these phenomena occur?' The prevailing analytical worldview attempts to address both, 'How?' and 'Why?'... The true scientific-philosophical worldview strives to respond, to the extent possible, not only to the questions of 'how and why,' but also to the questions: 'To what extent and for what purpose?' (Bugaev, 1898, p. 715)[1]

Bugaev viewed the process of cognition as a collection of infinitely many discrete parts into a coherent whole, with a conscious understanding of how these discrete parts transition into one another, and contribute to the overall outcome (hence the 'to what end' and 'for what purpose' questions). Based on this, Bugaev formulated his approach to teaching mathematics consisting of three components: theory, calculation mechanisms, and apply theory to solving practical problems. In his 1869 speech "Mathematics as a Scientific and Pedagogical Tool," he explains:

> Theory acts in a developing way on thought, forcing one to rethink in a systematic form what humanity has discovered after a long series of efforts… The mechanism of calculation is the language by which the mathematician expresses his ideas, poses, and solves his questions. Not being able to master the mechanism means not being able to master this great tool of our civilization, not being able to express one's thoughts in the mathematical language… Finally, the application of theoretical principles and the developed mechanism to solving practical problems constitutes the third most important aspect of the pedagogical influence of mathematics on the development of intellectual abilities... The educational power of mathematical exercises in solving various problems is manifested in the development of independence. This aspect in the best way completes the influence of mathematics on the development of reasoning and is the best measure and means for developing mathematical abilities. In teaching, these three aspects should follow in the very order in which we have presented them, and only their complete combination has the most beneficial effect on the mind (as cited in Kolyagin & Savina, p. 221)

The practical realization of these principles and concepts was Bugaev's publication of school textbooks over the next two decades. Beginning the series with his *Arithmetic of Whole Numbers* textbook, Bugaev executed the foundation of his educational views–integrating the parts of school courses with each other, providing a holistic view of mathematics and its connections with the overall process of cognition and other related fields.

Bugaev's Textbook

Arithmetic of Whole Numbers is part one of Bugaev's textbook series on arithmetic, with the second part being *Arithmetic of Fractional Numbers*. The textbook consists of two parts: The *rukovodstvo* (manual guide), first attempted to be published in 1874, and a *zadachnik* (problem book), which followed later in 1875. Bugaev's textbook was not easily published–it was rejected in 1874 by the Ministry of Public Education. Below is a direct quote from a letter written to Bugaev by a notable mathematician of the time, Aleksandr Korkin, expressing his opinion about the rejection of the textbook:

1 All translations from Russian are by the author.

> In Russian pedagogical literature of today's time, guides by people competent in their subject have rarely started to appear… Imagine my surprise when I learned that it was not approved by Dmitriev as a guide at the gymnasia of the Ministry of Public Education. Having read his review, I was outraged by the remarks, from which it all consists. Some of them cannot be explained by the mere ignorance of Dmitriev-the-mathematician; it is evidently a desire to disapprove the book at all costs (as cited in Kolyagin & Savina, 2009, p. 165).

It was finally published in 1875 as a manual and problem book together. Bugaev's arithmetic books proved to be successful for decades after. Although the exact number of editions is unknown, the books were reprinted at least dozens of times, with editions 11 and 12 being the most popular, and approved by the Ministry of Public Education for use in secondary schools (Kolyagin & Savina, 2009).

Although both parts of the book could be used independently and written by Bugaev in separate years, they are designed to be absolutely interconnected as parts of an organic whole, in accordance with Bugaev's conceptual vision. Bugaev believed that it is more useful to collect practice problems in dedicated problem books rather than scatter them throughout a textbook, since mixing problems with the theory itself disrupts the integrity and logical flow of the material (Bugaev, 1881). Conceptually, the manual clearly corresponds to the first two elements of Bugaev's approach mentioned earlier–theory and calculation mechanisms. In contrast, the problem book corresponds to the third component of application of theory to solving practical problems.

Manual Guide

Bugaev presents the manual as a guide for students entering gymnasia at age 10. Before enrolling in the first year of gymnasia, these children have already acquired basic arithmetic skills, such as performing simple operations and knowing the multiplication table. He also claims that it is not a self-study textbook in any way, nor is it his intention to replace the teacher's role with this manual. Rather, he focused on providing a clear, concise, and systematic presentation of arithmetic truths (Bugaev, 1898). The topics Bugaev includes in the manual are: numeration/counting; basic arithmetic operations (addition, subtraction, multiplication, and division); checking operation results; order of operations (with parentheses); units, conversions, and operations with units; applications of arithmetic operations to problem solving; historical details on numeration; and counting in various bases.

With Bugaev's goal of providing students with the whole, systematic approach of arithmetic truths, Bugaev guides students through their process of cognition, and helps them to answer the questions, 'To What Extent?' and 'For What Purpose?' This is reflected in Bugaev's strictly organized structure of writing in the manual–definitions of terms and concepts are introduced in each chapter, followed by informal explanations resembling those that a student would encounter in a classroom setting from the words of a teacher. For example, in Chapter 2, Bugaev explains the notion of counting and numeration, including an informal proof of why the set of whole numbers is an infinite set:

> **Counting:** When counting objects, we enumerate whole numbers in sequential order. In doing so, we move from one whole number to the next larger number by adding one each time.
>
> **The number of whole numbers:** There are an infinite number of whole numbers because no matter how large a whole number is, it is always possible to obtain the next larger whole number by adding one to it. Each whole number must have a specific name and symbol so that it can be distinguished from other numbers both verbally and in writing…
>
> Whole numbers are an infinite set. If there were a separate word for each whole number, it would be impossible to remember them. This difficulty is avoided by using special methods of verbal expression, which constitutes the subject of verbal numeration (Bugaev, 1898, p. 6).

Another example of his textbook style can be taken from the last chapter on problem solving:

> **Problem 20.** A certain person, having a capital of 8998 rubles, bought 15 dessiatins [unit of measure] of arable land at 125 rubles each, 37 dessiatins of meadow at 112 rubles each, and 5 horses at 147 rubles each. With all the remaining money, he bought forest land at 132 rubles per dessiatin. How many dessiatins of forest land did he buy? (Bugaev, 1898, p. 116)

After providing the student with the problem above, Bugaev guides them through a plan, or algorithm, of problem solving the students should take when attempting to approach the problem:

> **Problem Composition.** It is easy to determine the composition of this problem. Our complex problem breaks down into the following 6 simple problems, among which:

The first problem determines how much he paid for the arable land and is solved by multiplication.

The second problem determines how much he paid for the meadow and is solved by multiplication.

The third problem determines how much he paid for the horses and is solved by multiplication.

The fourth problem determines how much money he spent on all these purchases and is solved by addition.

The fifth problem determines how much money he had left after these purchases and is solved by subtraction.

The sixth problem determines how many dessiatins of forest land he bought with the remaining money and is solved by division (Bugaev, 1898, p. 117).

Although Bugaev didn't intend to replace school teaching with his manual guide, the provided guidance to problem solving encourages independence, which Bugaev emphasized was a significantly important skill in his 1899 address titled, "On the question of secondary school": "In my opinion, a person's ability to independently, actively, and energetically acquire knowledge should be valued more highly than the knowledge itself" (as cited in Kolyagin & Savina, 2009, p. 253.) As such, Bugaev also includes a chapter on checking operation results, in which he encourages the students to confirm that they got the right answer and increases their confidence by providing numerous methods of checking their work with the four operations. He claims that it is not enough to simply repeat the operation another time, but that "our confidence increases if we verify the correctness of a result by another method" (Bugaev 1898, p. 63). Bugaev then introduces two distinct ways of checking the completed operation: one using the same operation but in a different order, and the other using the opposite operation (see Figures 1 and 2).

Bugaev dedicates the concluding chapter (appendix) of the manual to exploring the historical evolution of counting and numeration systems. He places considerable emphasis on the foundational component of theory, which systematically enriches understanding by revisiting the significant discoveries made by humanity throughout history. Bugaev provides a detailed examination of the development and interconnections of numeration methods in Chinese, Finnish, Greek, Roman, and Church Slavonic languages, tracing their dissemination across Europe (Bugaev, 1898). See Figure 3.

Figure 1

Checking an addition problem by using the same operation (Bugaev, 1898, pp. 63-64)

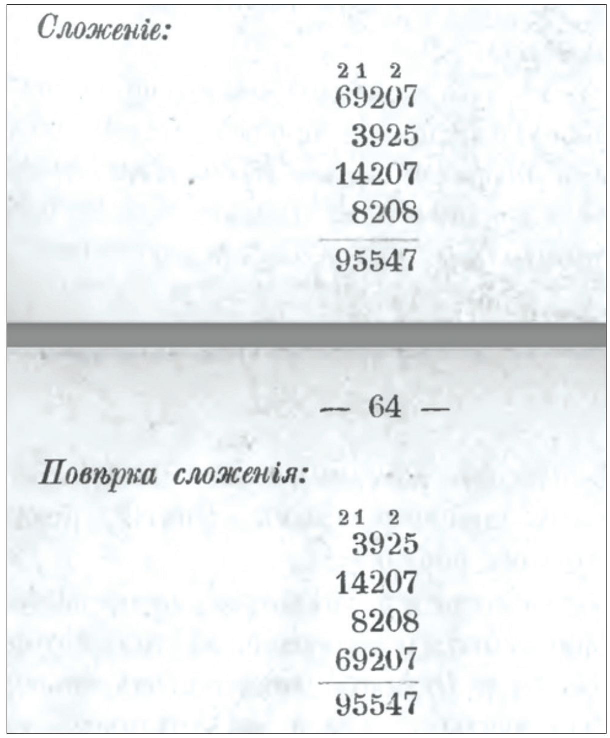
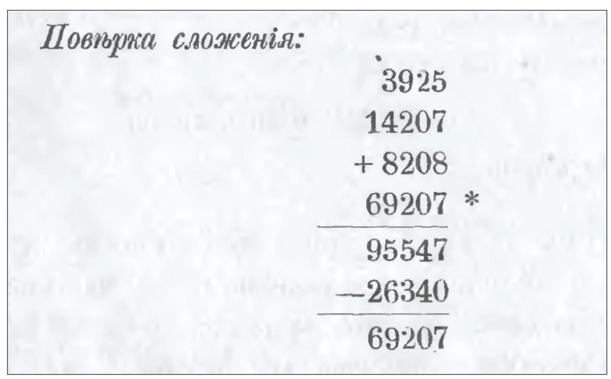


Figure 2

Checking an addition problem by using the inverse operation (Bugaev, 1898, pp. 66)



Problem Book

Bugaev's problem book is structured to match the manual completely, with each chapter in the problem book corresponding to a chapter in the manual and providing two sections–questions and problems–which complement the theory provided in the manual. According to Bugaev, the questions in each chapter emphasize aspects of the theory that "compose their systematic

Figure 3

Comparison of different number systems (Bugaev, 1898, p. 122)

Числа	Греческія,	Славянскія.	Числа	Греческія,	Славянскія.
1	α	а҃	60	ξ	ѯ҃
2	β	в҃	70	ο	о҃
3	γ	г҃	80	π	п҃
4	δ	д҃	90	η	ч҃
5	ε	є҃	100	ϱ	р҃
6	ς	ѕ҃	200	σ	с҃
7	ζ	з҃	300	τ	т҃
8	η	и҃	400	υ	у҃
9	ϑ	ѳ҃	500	φ	ф҃
10	ι	і҃	600	χ	х҃
20	κ	к҃	700	ψ	ѱ҃
30	λ	л҃	800	ω	ѿ҃
40	μ	м҃	900	ϡ	ц҃
50	ν	н҃			

understanding of arithmetic as a subject" (Bugaev, 1876, Preface). These questions strive to make students think deeper about the concepts that they have read about in the manual, while the problems that follow are meant for practical application problem solving. In the introduction to the problem book, Bugaev claims that the problems in the book are split into two types: Those with a goal of developing skills in calculation, and those with a goal of reaching an understanding and meaning of arithmetic operations and their applications. In this way, rather than just blindly applying learned skills to solve elementary arithmetic problems, Bugaev encourages students to consciously apply the learned concepts to, again, be able to answer the two main questions 'To What Extent?' and 'For What Purpose?'

Let's give a few examples of questions and problems from the first chapter (Foundational arithmetic concepts).

<u>Questions:</u>

-How are numbers classified by their relationship with their unit?

-What relationship exists between a whole number and counting?

<u>Problems:</u>

-Name some whole numbers with units.

-Place the following whole and fractional numbers into a sequence of numbers: five, one fourth, twenty, one seventh, three fourths, eight, eleven, twenty, three eighths, one eighth (Bugaev, 1876 pp. 1-2).

Note that as mentioned previously, although Bugaev dedicated this textbook entirely to whole numbers, he includes fractional concepts in the practice problems from the subsequent step of mathematics to achieve the wholeness and continuity of mathematics teaching.

Additionally, rather than practical applications of mathematics such as sales and agriculture, which were common in math textbooks at the time, Bugaev's word problems throughout the problem book can be categorized under scientific fields, such as astronomy, geography, history, and biology (Gavrilova, 2017). In his

1869 speech, Bugaev said that "...the degree of development of mathematical deduction primarily determines the nature and level of our knowledge of the external world" (as cited in Kolyagin & Savina, 2009, p. 216). Viewing the seemingly simple arithmetic concepts through various drastically different scientific lenses demonstrates the role of mathematics and its relationship to science, as well as motivates students to think and learn about the universe through mathematics, aiding their process of cognition. As an example, below are a few problems from Chapter 2 (Numeration):

Problems:

-Write the italicized words in following statements as digits:

(a) There are *eight* major planets.
(b) The earth has *one* satellite.
(c) There are *five* continents.
(d) There are *four* cardinal directions.
(e) Russia has *two* capitals.
(f) A week has *seven* days.
(g) There are *nine* significant digits.
(h) There are *six* male gymnasiums in Moscow.

-Write the italicized word in the following statement as a number: A child human heart beats *one hundred and forty* times a minute, and an elderly human heart beats sixty times a minute (Bugaev 1876, pp. 5, 8).

a few examples from Chapter 3 (Basic arithmetic operations with whole numbers):

Problems:

-Blood circulates in the body 350 times in one day. How many full times will blood pass through the body in 3 weeks?

-Sound travels 1107 feet in one second. Thunder will be heard 13 seconds after seeing lightning. How far from us does the thunder occur? (Bugaev, 1876, pp. 23, 25)

Through solving these problems, the student engages with various scientific fields of knowledge. It can help to consciously choose a life's career. In this way, Bugaev attempted to address a major problem of secondary education that he formulated below in his 1899 article "On the question of secondary school":

> ...gradually, they did not know where to turn, and, barely finishing gymnasia, they filled the universities. Their low level of intellectual and moral development adversely affected the entire course of their university education. In the gymnasium, they often developed a distaste for all sciences and cared only about obtaining a university diploma. From them emerged poor doctors, lawyers, and educators (as cited in Kolyagin & Savina, 2009, p. 249).

The problem book is majorly composed of challenging problems; to effectively solve challenges, it is crucial for students to fluently 'speak' the language of mathematics, both in symbols and words.

Discussion and Conclusion

Nikolai V. Bugaev wrote his arithmetic textbooks during a time of significant transition from elite to mass education. As Russia modernized, the demand for accessible and rigorous mathematics education grew, and Bugaev's textbooks attempted to meet these evolving needs. While his works did not achieve the same widespread adoption as others, such as the arithmetic textbooks by A.P Kiselev, or those by A.F Malinin and K. P Burenin, they were, nonetheless, innovative for their time, emphasizing the integration of theoretical knowledge, practical application, and cognitive development. Bugaev's philosophy of teaching mathematics to develop independent problem solving and logical reasoning remains relevant today. In an era where critical thinking and analytical skills are increasingly valued, Bugaev's educational methods offer insights into the ongoing need for a holistic approach to mathematics education. His textbook serves as a reminder that mathematics education is not merely about computation, but also about shaping intellectual capacity and adaptability, skills that continue to be essential in modern education systems.

Bugaev's textbooks remain an example of a textbook written by a renowned mathematician who thought about both the philosophical foundations of knowledge and the pedagogical aspects of its acquisition. The philosophical studies of the Moscow School of Mathematics have attracted attention globally (e.g., Svetlikova, 2013), where Bugaev's influence is acknowledged not only in mathematics but also in literature (as the prototype of the heroes of all his son Andrei Bely's novels), and politics. In considering this complex and multifaceted cultural phenomenon, Bugaev's pedagogical contributions–both theoretical and practical–must not be overlooked, as they influenced the education of hundreds, if not thousands, of students.

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