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NOTES FROM THE FIELD

Beyond Memorization: Advocating Derivation and Proof in School Mathematics

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ABSTRACT This paper advocates for the teaching and emphasis of derivation and proof in school mathematics as a means to foster deeper conceptual understanding and reduce overreliance on rote memorization. By examining the derivation of key geometric formulas-such as the volumes of a cylinder, cone, prism, and pyramid, as well as the shortest distance between two points on a sphere in latitude and longitude-it demonstrates how students can engage meaningfully with mathematical ideas. Through logical reasoning, spatial visualization, and mathematical connections, derivation empowers learners to appreciate mathematics as a coherent and purposeful discipline. The paper calls for an instructional shift toward reasoning-based learning in school curricula to cultivate critical thinking and lasting comprehension.

KEYWORDS *Conceptual Understanding, Derivation, Proof, School Mathematics*

Introduction

In school mathematics teaching, there is an alarming tendency toward emphasizing rote memorization of formulas and procedures rather than fostering genuine understanding through derivation and proof. This prevailing pedagogical approach has raised significant concerns among educators and researchers regarding its impact on students' deep conceptual grasp, critical thinking skills, and long-term retention of mathematical knowledge (Boaler, 2016). Formulas for volumes of solids such as cylinders, cones, prisms, and pyramids, as well as geometric concepts like the shortest distance between two points on a sphere, are often presented as facts to be memorized rather than understood through logical reasoning or derivation. The resulting disconnect reduces mathematics to a collection of disconnected rules, hindering students' ability to apply their knowledge flexibly or appreciate the underlying structure of mathematics (Schoenfeld, 2014).

Derivation and proof are foundational pillars of mathematics. They provide learners with the tools to understand why formulas and relationships hold true, not merely that they exist. For instance, understanding that the volume of a cylinder, expressed as $V=\pi r^2 h$, can be logically deduced from the area of its circular base and its height helps students recognize how mathematical formulas are not merely to be memorized but derived through structured reasoning, a process central to the development of deductive proof skills (Miyakawa, Fujita, & Jones, 2017). Similarly, the volume of a cone, which is exactly one-third that of a cylinder with the same base and height, emerges from geometric reasoning or Cavalieri's Principle, rather than arbitrary acceptance. Cavalieri's Principle states: If two solids have equal heights, and if the areas of their corresponding cross-sections taken at equal distances from their respective bases are always equal, then the two solids have equal volumes. Using this principle, one can show that a cone has one-third the volume of a cylinder. By comparing the cone with a cylinder and a carefully constructed prism, students can visualize that the areas of cross-sections at each height differ by a factor of three, thereby justifying the one-third relationship in volume. Such derivations cultivate deeper mathematical insight by bridging intuitive reasoning with formal justification, thereby enabling learners to reconstruct or modify formulas through logical argumentation, an approach that reflects the central role of proof in linking mathematics as practiced by mathematicians to mathematics as taught in schools (Rocha, 2019).

It is, however, important to specify the grade levels where such proofs and derivations are appropriate. For example, derivations involving prisms and cylinders (using cross-sections or stacking) can be introduced as early as grades 6–8, since they require only basic geometry and arithmetic. Proofs involving cones and pyramids via Cavalieri's Principle are more appropriate for grades 8–10, when students are mature enough to follow abstract reasoning. Advanced topics such as the derivation of the great-circle distance on a sphere, which relies on trigonometry, are best suited for grades 11 and 12, when students have developed sufficient algebraic and trigonometric background. Explicitly linking examples to grade levels ensures that derivation and proof are introduced progressively, making them accessible and meaningful to learners at different stages of development.

In many systems and pedagogical contexts, however, time constraints, curriculum demands, and examination pressures compel teachers to prioritize formula memorization over reasoning. The consequence is a superficial grasp of mathematics where students can reproduce formulas but struggle to explain or derive them. This not only limits students' engagement in the active roles of exploring, conjecturing, and justifying through proof but also undermines their confidence and sustained interest in mathematics (Bleiler-Baxter & Pair, 2017). Moreover, teachers often resist the integration of derivation and proof because they fear it will reduce the time available for procedural drill, an emphasis reinforced by high-stakes examinations (Hiebert & Grouws, 2007). Studies reveal that while conceptual teaching improves long-term achievement, many teachers prefer the certainty of procedural coverage to meet assessment demands (Thompson & Senk, 2012). Addressing these concerns requires practical strategies for integration. One approach is to embed short, intuitive derivations within existing lessons, for instance, spending five minutes showing why the cylinder's volume equals "base area × height" through stacking, before assigning routine practice. Another strategy is the use of visual or hands-on demonstrations, such as water-filling experiments to compare cone and cylinder volumes, which require little extra class time yet leave lasting impressions. Furthermore, schools could align derivation-focused instruction with curriculum objectives, ensuring that it supports, rather than competes with, exam preparation. Technology-based tools like dynamic geometry software also allow quick and interactive visualization of proofs, helping teachers overcome time and resource limitations (Stylianides & Stylianides, 2009).

Mathematical topics such as the volume of prisms and pyramids provide fertile ground for teaching derivation and proof in school mathematics. The volume of a prism, found by multiplying the base area by height, can be understood by considering the prism as a stack of congruent cross-sectional areas along its height. Likewise, the formula for the volume of a pyramid, which is one-third the volume of a prism with the same base and height, can be illuminated through spatial reasoning or geometric dissection, thus demystifying why the factor of one-third appears and how volume relates to base and height in different solids (Mason, Graham, & Johnston-Wilder, 2005). These examples are particularly suitable for junior secondary levels (Grades 7–9), when students are transitioning from concrete to abstract reasoning. Another illustrative example lies in spherical geometry, particularly the determination of the shortest distance between two points on a sphere, expressed in terms of their latitude and longitude coordinates. This concept, fundamental in navigation and geography, is often introduced in a formulaic manner without deriving the great-circle distance formula from basic geometric principles. When students are guided through the reasoning behind the spherical law of cosines or the haversine formula, they develop a concrete understanding of how curvature influences distance and why Euclidean notions of straight lines do not apply on curved surfaces (Feeman & Green, 2015). Such derivations are better suited for upper secondary students (Grades 11–12), where trigonometric concepts are already part of the curriculum.

Recent research underscores the benefits of integrating derivation and proof in mathematics education. Studies indicate that students exposed to reasoning-focused instruction demonstrate higher achievement, better problem-solving abilities, and greater motivation (Stylianides & Stylianides, 2009; Boaler, 2016). Furthermore, national curriculum reforms in countries like Singapore and Finland emphasize conceptual understanding and mathematical reasoning, suggesting a growing consensus on the importance of proof and derivation in school mathematics (Ng & Widjaja, 2015). Despite these developments, many educational systems lag behind, continuing to undervalue derivation and proof in favour of procedural fluency. This paper therefore advocates a renewed emphasis on teaching derivation and proof within school mathematics, arguing that such an approach nurtures mathematical thinking, enhances understanding, and equips students with skills essential for further study and everyday life. By explicitly clarifying which grade levels suit which derivations, providing clear explanations of concepts like Cavalieri's Principle, and offering strategies for practical classroom integration, the study responds to the concerns of both researchers and practitioners. By focusing on classical yet foundational examples, volumes of common solids and shortest distances on spheres, the study illustrates how derivation and proof can be effectively integrated into the curriculum without overwhelming teachers or students.

Some Illustrations

In this section, the derivations and proofs in school mathematics are illustrated.

1. Derivation of volume of a cone formula

The derivation of the volume of a cone formula may not be obvious in school mathematics like that of cylinder. A cylinder can be visualized as a stack of identical circular discs. The volume of a cylinder is obtained as Volume = base area (circle) × height, which gives $V=\pi r^2 h$ This is clear enough. Imagine filling a cone with water and pouring it into a cylinder with the same base and height. It takes exactly 3 full cones to fill the cylinder. But the result comes from solid geometry developed by Archimedes (circa 287–212 BC) and formalized by Cavalieri's Principle (formulated by Bonaventura Cavalieri in the 17th century) which states: If two solids have the same height and the same cross-sectional area at every level (parallel to the base), then they have the same volume. There is also a converse: If the cross-sectional areas are in a constant ratio at every height, then the volumes are in the same ratio.

Let us derive the volume of a cone using integration. Take a right circular cone with height h and base radius r. Place it so that the tip is at the origin and the base is at $x=h$. The equation of the slant side (a straight line from (0,0) to (h,r)) is: $y= (h/r)x$. Rotate this line around the x-axis to form the cone. Using the disk method (see Figure 1):

Figure 1

Disk Method

$$V = \pi \int_0^h y^2 dx = \pi \int_0^h \left(\frac{r}{h}x\right)^2 dx = \pi \int_0^h \frac{r^2}{h^2} x^2 dx = \pi \frac{r^2}{h^2} \int_0^h x^2 dx = \pi \frac{r^2}{h^2} \left[\frac{x^3}{3}\right]_0^h = \pi \frac{r^2}{h^2} \left[\frac{h^3}{3}\right] = \frac{1}{3}\pi r^2 h$$

So, whether you understand it geometrically or through calculus, the volume of a cone is:

$V=\frac{1}{3}\pi r^2 h$

Here is a clear and logical derivation of the formula for the volume of a cone without using calculus, using geometric reasoning. You can derive the volume of a cone geometrically by knowing the formula for a cylinder and observing that a cone with the same base and height fits 3 times into the cylinder based on Cavalieri's Principle. That is:

$$\text{Volume of cone } (V_{cone})=k\times\text{Volume of cylinder}(V_{cylinder}),$$

$$k = \text{constant of proportionality}, 0<k<1$$

$$\Longrightarrow \quad kV_{cylinder} + kV_{cylinder} + kV_{cylinder} = V_{cylinder}$$

$$\Longrightarrow \quad 3kV_{cylinder} = V_{cylinder} \Longrightarrow 3k = \Longrightarrow k=\tfrac{1}{3}$$

$$\therefore \quad V_{cylinder} = \tfrac{1}{3}\pi r^2 h$$

Therefore, the volume of a cylinder is three times the volume of a cone with the same base and height. Just as a cone occupies exactly one-third the volume of a cylinder with the same base and height, a pyramid also occupies exactly one-third the volume of a prism with the same base area and height. This analogy helps students understand the volume relationship without needing calculus.

2. Proving the shortest distance formula on a sphere (the Earth) using longitude and latitude in school mathematics

We want to find the formula for the shortest distance between two places on the Earth's surface using their longitudes and a common latitude. The formula is:

$$\boldsymbol{D_s = \frac{2\sin^{-1}(\cos\alpha \sin\frac{\theta}{2})}{360} \times 2\pi R}$$

where:

D_S is the shortest distance between the two points,

θ is the difference in longitude (in degrees),

α is the common latitude,

R is the radius of the Earth.

But where does this formula come from? Many mathematics textbooks state the formula without explanation (Obasi, 2015). Let us understand how it is derived, as presented in Obasi (2015), and replicate it here for easy reference.

Proof

Understanding the idea. On a flat surface, the shortest distance between two points is a straight line. On a curved surface like the Earth (which is almost a sphere), the shortest path between two points is called a great-circle distance. To understand this, imagine the Earth as a circle and draw two points A and B on the same latitude, but different longitudes. Connect them with a chord (a straight line through the circle). Let the angle between these two points be θ (in degrees). The chord forms part of a sector of a circle. Using the sector formula:

$$D_s = \frac{\vartheta}{360} \times 2\pi R \qquad (1)$$

To determine ϑ, since at small θ, the length of Arc equals the length of the Chord. The length of a chord of a circle is:

$$L = 2r \sin\frac{\theta}{2} \qquad (2)$$

where θ is the angle subtended by the Chord. Similarly, ϑ is the angle subtended by the shortest distance, which is given by

$$L_2 = 2R \sin\frac{\vartheta}{2} \qquad (3)$$

But $r = R\cos\alpha$, then equation (2) becomes

$$L = 2R\cos\alpha \sin\frac{\theta}{2} \qquad (4)$$

Since Chord of a circle is uniform, therefore equation (3) is equal to equation (4), i.e.

$$2R \sin\frac{\vartheta}{2} = 2R\cos\alpha \sin\frac{\theta}{2}$$

$$\sin\frac{\vartheta}{2} = \cos\alpha \sin\frac{\theta}{2}$$

$$\frac{\vartheta}{2} = \sin^{-1}(\cos\alpha \sin\frac{\theta}{2})$$

$$\therefore \vartheta = \sin^{-1}(\cos\alpha \sin\frac{\theta}{2})$$

Therefore, $\boldsymbol{D_s = \frac{2\sin^{-1}(\cos\alpha\sin\frac{\theta}{2})}{360} \times 2\pi R}$

And this is the shortest distance formula between two points with the same latitude but different longitudes on a sphere like the Earth. This proof shows how the shortest distance formula is derived from basic geometry, not just memorized. Understanding the why behind formulas helps you become a creative and confident problem solver—just like the great mathematicians. Let this motivate you to go beyond formulas and think about the ideas behind them.

3. Proof of why we invert the second fraction when dividing

Why do we invert when dividing two fractions? Teachers often instruct students to "invert the second fraction and multiply" when dividing two fractions. While this rule is mathematically correct, it is frequently taught without explanation, leaving students to accept it as a mysterious trick—what might be called *mathemagic*. However, mathematics should make sense, not just work by rules. The following logical proof explains why the inversion step works, helping students understand the reasoning behind the rule rather than memorizing it blindly.

Let's divide:

$$\frac{a}{b} \div \frac{c}{d}$$

This means: "How many times does $\frac{c}{d}$ fit into $\frac{a}{b}$?" Let x be the answer:

$$\frac{a}{b} = x \times \frac{c}{d}$$

Multiply both sides by $\frac{c}{d}$ to isolate x: $x = \frac{a}{b} \times \frac{d}{c}$

Therefore:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c}$$

This logical proof justifies the "invert and multiply" rule.

Conclusion

The overreliance on memorization in school mathematics has created a generation of learners who often lack genuine understanding of mathematical concepts and struggle to apply them flexibly. This paper has demonstrated, through the derivation of formulas for volumes of common geometric solids, cylinders, cones, prisms, and pyramids, as well as the shortest distance between two points on a sphere, that deep mathematical understanding is achievable when students are guided through reasoning and proof. These derivations, when introduced appropriately in school curricula, help students see mathematics not as a set of disconnected rules, but as an elegant and logical system grounded in relationships and patterns. Advocating for derivation and proof in school mathematics is not just a call for curriculum reform, it is a call to transform how students experience and internalize mathematics. When learners are given the tools and time to explore why a formula works, they gain confidence, build critical thinking skills, and develop a more lasting appreciation for the subject. Teachers, curriculum developers, and policymakers must therefore prioritize reasoning, exploration, and derivation as core components of mathematics instruction. Only then can mathematics teaching move beyond mechanical performance and toward meaningful, enduring understanding.

Suggestions

The following suggestions are made:

1. Schools should integrate intuitive and visual derivations of formulas (such as prism and cylinder volumes) at the junior secondary level (Grades 6–8), while reserving more abstract derivations (such as Cavalieri's Principle for cones and pyramids or spherical trigonometry for great-circle distances) for senior secondary levels (Grades 9–12).
2. Teachers can embed short derivations within regular lessons, use hands-on demonstrations (e.g., water-filling experiments for cone and cylinder volumes), and employ dynamic geometry software to visually illustrate proofs without significantly reducing time for procedural practice.
3. Curriculum designers and examination boards should incorporate reasoning-based questions alongside procedural ones, so that teachers are motivated to balance formula memorization with proof, fostering both conceptual understanding and exam readiness.

References

Bleiler-Baxter, S. & Pair, J. (2017). Engaging students in roles of proof. *J. Math. Behav. 47*, 16–34. https://doi.org/10.1016/j.jmathb.2017.05.005

Boaler, J. (2016). *Mathematical mindsets: Unleashing students' potential through creative math, inspiring messages and innovative teaching*. Jossey-Bass. https://doi.org/10.1080/14794802.2016.1237374

Feeman, T. G., & Green, J. (2015). *Portraits of the Earth: A mathematician looks at maps* (2nd ed.). American Mathematical Society. https://doi.org/10.1090/mawrld/018

Mason, J., Graham, A., & Johnston-Wilder, S. (2005). *Developing thinking in algebra.* SAGE Publications.

Ng, S. F., & Widjaja, W. (2015). Teaching with variation: A Chinese way of promoting deep understanding of mathematics. In C. Sun, B. Kaur, & J. K. L. Lee (Eds.), *Mathematics education in Singapore* (pp. 103–117). Springer. https://doi.org/10.1007/978-981-13-3573-0

Obasi, C. (2015). Enhancing students' creativity in mathematics through proof: Some random illustrations. *Journal of Issues on Mathematics, 17*(1), 68-77.

Rocha, H. (2019). Mathematical proof: from mathematics to school mathematics. *Philos Trans A Math Phys Eng Sci.*, 377(2140):20180045. http://dx.doi.org/10.1098/rsta.2018.0045.

Schoenfeld, A. H. (2014). What makes for powerful classrooms, and how can we support teachers in creating them? *Educational Researcher, 43*(8), 404–412. https://doi.org/10.3102/0013189X14554450

Stylianides, A. J. (2007). Proof and proving in school mathematics. *Journal for Research in Mathematics Education, 38*(3), 289–321. https://doi.org/10.2307/30034869

Stylianides, G. J., & Stylianides, A. J. (2009). Facilitating the transition from empirical arguments to proof. *Journal for Research in Mathematics Education, 40*(3), 314–352. https://doi.org/10.5951/jresematheduc.40.3.0314