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Making Connections in Mathematics Education

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After Presenting Multiple Solution Strategies, What's Next? Examining the Mathematical Connections Made by Preservice Teachers

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ABSTRACT When teaching through problem solving, effective mathematics teachers need to lead discussions that assist students in making connections between different solution strategies. However, while teaching a methods course for preservice teachers (PSTs), we noticed that after solving a problem and presenting various solution strategies, many PSTs seemed lost on how to proceed with the mathematics lesson. To address this issue, we designed an action research study where we implemented Smith and Stein's (2011) five practices for orchestrating productive classroom discussions, and focused our attention on the fifth practice, making connections. Specifically, we designed an instructional intervention to examine the type of connections made by PSTs and how these connections changed as the course progressed to aid PSTs' connection making skills. We identified three types of connections made by PSTs: superficial knowledge connections, procedural knowledge connections, and conceptual knowledge connections. Additionally, we observed a decrease in the amount of superficial knowledge connections and an increase in the amounts of procedural knowledge connections and conceptual knowledge connections made by PSTs throughout the course.

KEYWORDS *connections, conceptual, mathematics, mathematics education, preservice teachers, procedural; teacher education*

Introduction

Making connections is central to learning in mathematics (Association of Mathematics Teacher Educators [AMTE], 2017; Bingolbali & Cusknn, 2016; National Council of Teachers of Mathematics [NCTM], 2000). According to the AMTE (2017) standards, it is the teachers' responsibility to lead effective discussions that draw out important mathematical connections from students; however, past research has shown that leading such discussions is particularly difficult for pre-service teachers (PSTs) (Ball, 1993; Lampert, 2001; Stein, Engle, Smith, & Hughes, 2008). Novice teachers do not have a reservoir of mathematical knowledge that can be used to identify connections, particularly at the spur of the moment (Schoenfeld, 1998). Furthermore, many PSTs learned mathematics in classrooms where connections were not emphasized; therefore, they may struggle to make connections and may not understand the important role of facilitating discussions that draw out connections to support student learning. Thus, aiding PSTs in making connections and helping them understand the types of connections that can be used to promote mathematical learning through their teaching must be a vital piece of undergraduate mathematics education courses.

Action Research

After teaching an undergraduate elementary mathematics content and methods course, we noticed a disconnect between our expectations about how PSTs should use connections and what we saw PSTs demonstrate during classroom discussions, in their reflective writing, and while teaching practice lessons in front of their classmates. To address this disconnect, we designed an action research project. Action research is a reflective process led by teachers in their classrooms (Patthey & Thomas-Spiegel, 2013). In this practitioner-centered approach, the teacher designs an instructional intervention for the classroom and evaluates its impact on students (Somekh & Zichner, 2009). In the current project, we created a new curriculum designed to develop PSTs' ability to use connections effectively in the elementary mathematics classroom. Our main goals were to investigate the types of connections PSTs made throughout the course and to evaluate whether PSTs made more effective connections as the course progressed.

Action research serves as a systematic way to improve the practice of educators, especially through its personal and reflective nature, by providing teachers with a method to improve the critical areas they choose to work on (Somekh & Zichner, 2009; Patthey & Thomas-Spiegel, 2013). Not only did we choose action research to help PSTs develop ways to make mathematical connections in problem solving, but also to become better educators. We wanted to learn from the collected data and use this information to further improve our teaching practices. This speaks to the continuous cycle of action research, where constant reflecting, monitoring and modifying are necessary for improvement (Patthey & Thomas-Spiegel, 2013). To engage in this reflective process, we conducted a making-connections activity in class, collected student work, read and analyzed responses, drew conclusions about the connections made, and used this information to construct a new activity for the next class.

We acted as instructors, observers, and participants in this study. We were the instructors, designing and grading all the assignments as the semester progressed. We acted as observers as we took field notes while PSTs presented mathematics problems and role-played different classroom scenarios. Finally, we acted as participants when we provided feedback and facilitated discussions. Because action research is also used as professional development (Oolbekkink-Marchand, Van der Steen & Nijveldt, 2014), playing the roles of observer and participant allowed us to examine the classroom through new lenses. This assisted our overall goal of becoming better educators through action research.

Problem-based Instruction

When using problem-based instruction, students are given genuine problems to solve. That is, problems for which students have not been told specific solution strategies to use and do not perceive there is a single, correct way to solve the problems (Hiebert, et al., 1997). Students create strategies for solving a given problem by building from their prior knowledge. This often requires several attempts at various ways to solve the problem before coming up with a logical approach. After developing strategies, students present and discuss the various strategies that were created. By presenting their strategies, students develop a sense of ownership as they are encouraged to agree or disagree with their classmates to further their understanding of the topic (Chapin, O'Connor, & Anderson, 2009).

Pre-service teachers need a structure to follow in order to create an effective problem-based classroom. One pedagogical method teachers can implement in the classroom is Smith and Stein's (2011) *Five Practices for Orchestrating Productive Mathematics Discussions.* Smith and Stein's (2011) five practices are a set of skills designed to assist teachers who use problem-based instruction. They are described by Smith and Stein (2011) as:

1. Anticipating potential student responses to challenging mathematical tasks;
2. Monitoring students' as they create solutions to the tasks;
3. Selecting particular students to present their strategies for solving the problem;
4. Sequencing the student responses to be displayed in a specific order; and
5. Making connections between the different solution strategies presented to highlight key mathematical ideas.

The teachers' role in the fifth practice, making connections, is critical. To enhance learning, the teacher must identify which mathematical connections they would like to focus on, understand how these connections are related to their learning goals for the lesson, and use students' mathematical thinking to help make these connections explicit (Stein et al., 2008). In our experience of teaching mathematics methods courses, PSTs struggle during this critical phase of a problem-based lesson. To address this, we designed an intervention focused primarily on the last practice, making connections. Smith and Stein (2011) argued that making connections must be at the forefront of teachers' thinking throughout the five stages. For example, the decision about who to select to present solution strategies should be guided by the connections that the teacher would like to emphasize during the whole class discussion. Moreover, with practice, PSTs can steadily improve at facilitating whole classroom discussions focused on making connections, a necessity when unpacking cognitively challenging math problems (Stein et al., 2008). This is where students have an opportunity to further their thinking and develop deep mathematical understanding of the material.

Connections

A mathematical connection is defined by Mhlolo, Venkat and Schäfer (2012) as "a process of making or recognizing links between mathematical ideas" (p. 2). The brain is equipped with the ability to make connections (Caine & Caine, 1991), and this ability can be enhanced when teachers conduct lessons that emphasize, draw out, and formulize the connections inherent in the mathematics being studied. Tchoshanov (2011) found that student achievement increased significantly as teachers' knowledge of concepts and mathematical connections increased. Thus, it is important for teachers to have a deep understanding about mathematics concepts and the relationships that exist between concepts to engage in discourse and questioning that makes connections apparent to students.

Mathematical understanding has been described in terms of instrumental understanding and relational understanding (Skemp, 1976). Instrumental understanding is defined as the ability to carry out procedures and rules without knowing why they work. This understanding is related to procedural knowledge as it requires the use of formal language, symbolic mathematical notations, and rules for completing a mathematical task (Hiebert & Lefevre, 1986). Teachers and students have relied on their ability to use procedures and rules to demonstrate their mathematical understanding. However, instrumental understanding falls short of developing the skill of making connections as the procedures used for specific problems are not always generalizable to other problems (Rittle-Johnson, Siegler, & Alibali, 2001).

Skemp (1976) described relational understanding as understanding which engages students in knowledge about mathematical operations and the reasons for why they work. Further, relational understanding is associated with conceptual knowledge as it consists of an integrated network of mathematical concepts, where each piece of information is connected with other information to develop mathematical understanding (Bingolbali & Cusknn, 2016; Hiebert and Lefevre, 1986; Skemp, 1976). This type of understanding is highlighted in the process of learning mathematics and is constructed in classrooms that emphasize the skill of making connections between concepts and among different representations of a concept.

When teaching to build relational understanding, identifying mathematical connections between solution strategies becomes a central component. For example, as students engage in double-digit addition, teachers can highlight connections by using place value understanding (CCSSO, 2010). As shown in Figure 1, when solving the problem 25 plus 38, a student may use the split method, separating each number by place value.

$$\begin{array}{rc} & 20 + 5 \\ + & 30 + 8 \\ \hline & 50 + 13 \end{array}$$

Figure 1. Split method for solving double-digit addition.

The teacher can use this as an opportunity to demonstrate how the sum of five and eight, both in the ones place, equals 13, a number with one ten and three ones. Further, through discussion and questioning, PSTs can help students connect the split method solution strategy to the memorized procedure of carrying over the ones to the tens place. This leads to meaningful learning because students are able conceptualize the underlying reasons for why procedures work and identify relationships that exist between multiple representations.

Teachers need to have an understanding about the types of connections that exist and how these connections promote mathematical learning. Smith and Stein (2011) argued that making connections is the most challenging practice, as it calls for teachers to identify important connections and create questions that make those relationships explicit for students. Questions that elicit powerful connections build on students' solution strategies by drawing out relationships between various strategies or relationships that exist between strategies and big mathematical ideas. By asking targeted questions, teachers can push students to think deeply about important mathematics.

Although an essential role of the teacher in the problem-based classroom should involve making connections, we know very little about the types of connections PSTs make, and whether their ability to make connections can improve over time with targeted instruction. Therefore, we designed a study to investigate the following questions:

1. What types of connections do PSTs make?
2. How did the types of connections PSTs make change throughout the study?

Methodology and Data Analysis

Data for this manuscript were collected at an urban university in the Southeastern United States. The sample was composed of 18 third- and fourth-year students who were completing their bachelor's degree in education. The study was conducted in a mathematics education content and methods class required for all early childhood, elementary, and special education majors. This semester long course met twice a week for one hour and 15 minutes for 16 weeks. The first two authors were the instructors of the course and the third author had extensive experience teaching that course. All three authors met weekly to discuss students' writing and create problems to further PSTs' understanding of making connections.

To guide each lesson, we modeled Smith and Stein's (2011) five practices. First, the instructors met outside of class and created an elementary mathematics task and listed anticipated solution strategies informed by the literature (Levi & Empson, 2011; Van de Walle, Karp, & Bay-Williams, 2015). Then, in class, we followed two approaches to assess PSTs' connection making. The approach included presenting PSTs with the task, asking them to solve it on their own, and asking them to share their strategies in small groups. At this point, we modeled the practices of monitoring, selecting, and sequencing. Frequently we chose three students to share their solution strategies with the entire class. Once the chosen students shared their strategies, we led a discussion where we asked the following recurring questions:

1. What big mathematical idea can you address by using this problem?
2. What connections can you make between the strategies to highlight the big idea?

In alternate lessons, we provided PSTs with possible solution strategies and then asked them the two questions above. In this scenario, we placed the PSTs in the role of the teacher by asking them to decide how they would proceed in a classroom where the given answers appeared (Grossman, Hammerness, & McDonald, 2009). We asked the two questions above to help PSTs clarify their thinking and to help us assess their thinking. The first question was necessary to understand the learning goal PSTs were looking to emphasize as well as to help demonstrate the importance of keeping the goal of the lesson at the forefront. The responses to the second question uncovered the types of connections PSTs were making and was designed to narrow PSTs answers to focus on mathematical connections between strategies. This writing was designed to help PSTs clarify, and then, formalize their thinking about the mathematical connections in the solutions presented. The writing was also used by the authors as formative assessment providing information about PSTs understanding and thinking about connections.

Our data consisted of nine responses to the two recurring questions above for each PST in our sample. Each week, PSTs engaged in a problem-solving activity and completed the writing assessment. After each activity, we collected students' writing, removed any identifying information, and made photocopies to use during our weekly meetings. We returned the original copies to PSTs with feedback about the connections they discussed. During our weekly meetings, we read PSTs responses, identified common codes, and determined themes describing the types of connections PSTs were making (Marshall & Rossman, 2006; Taylor & Bogdan,1998). The information obtained during these meetings informed the development of the activity used in the following week's lesson. We continued this process throughout the semester, analyzing each new set of data, updating and refining our list of codes, and using this information as we designed the curriculum.

As we read through PSTs' responses to the first activity, we used content analysis, which focuses on identifying concepts within texts, and created a list of common phrases we observed throughout their writing (Carley, 1993). Then, two of the researchers coded the assignment using this list while looking for emergent codes (Taylor & Bogdan 1998). For example, in response to question two, a PST stated, "I would also explain to them how 6/9 simplifies to 2/3 so they can understand that both fractions are still equivalent (1). I would even take one of the drawings from Ana's picture and one from Ben's and shade in 2/3 and 6/9 so they can visually see they're equal (2)". We coded the first statement (1) as *describing the big idea* because the PST described equivalence and statement (2) as *multiple representations* because the PST pointed out that two different representations were equivalent to each other. Throughout the process, we also compared our coding to combine similar codes, delete infrequent codes, and write definitions for each code. For example, the code *preferred learning styles* and *variety* were combined to create the code *various strategies.* This process was repeated several times while concurrently informing our coding by reading research in mathematics education. When the codes were finalized, the data were coded again in their entirety. All disagreements were discussed at length until we arrived at the same conclusion. For example, during the first iteration, learning styles and variety emerged as separate codes. After further discussion, we concluded that in both codes PSTs were mentioning different strategies without connecting them, thus we combined these to create the code of *various strategies.*

Then, we carefully examined all codes to search for themes. When several codes described the same component within the data, we combined them to create an overarching theme. For example, the codes *same answer, naming a concept,* and *various strategies* were combined to create the theme *superficial knowledge connections.* Finally, each theme and code were defined (see Table 1). These themes are representative of the types of connections made by PSTs.

For our analysis of the first research question investigating the types of connections PSTs made, we analyzed the assessments associated with all 9 problem-solving activities. For the analysis of the second research question where we sought to investigate how the types of connections PSTs made changed throughout the study, we selected the assessments from three activities, one that took place at the beginning, middle, and end of the semester. We will briefly describe the problems that guided the three activities (see the Appendix for the full problems. The first activity involved solving the problem 70 minus 59. PSTs were presented with strategies involv-

Table 1
Codes with their descriptions and examples.

Code	Description	Student Example
Superficial Knowledge Connections		
Same answer	Emphasis on the numerical result being the same.	Sandra, Alice and Milagros used differing strategies and got the same answer.
Naming a concept	Stating the use of a concept without showing understanding.	The connections that I would address would be place value and how all of the examples kept place value in mind.
Various strategies	Mentioning a variety of solution strategies without connecting them.	Milagros does the problem very traditionally where as Alice breaks her problem down.
Procedural Knowledge Connections		
Step by step	Describing step by step the process used to get the answer.	I could also ask the students how many 3's go into 6, they would get 2. Then, ask them how many 3's go into 9 and they would get 3, therefore ⅔ = 6/9.
Operation used	Pointing out the same operation was used in both or more strategies.	All use subtracting in one form or another to get to the final answer.
Conceptual Knowledge Connections		
Describing the big idea	Students' understanding of general rules, facts, and definitions and using these appropriately to describe the big idea.	The big idea could be the commutative property. Connection is that all the numbers are grouped differently showing that you can switch the order in which its multiplied. For example, 6 x 3 to 3 x 6.
Knowing why a procedure works	Students' understanding of why a specific procedure works and the ability to apply it to solve math problems.	Ben's strategy is a more expanded version of Cal's because he drew out the 6 subs technically and divided it into 9 each time. Cal, on the other hand, automatically knew that he would have to use the drawing that he made 6 times for the 6 subs, so he skipped the addition and multiplied instead.
Multiple representations	Pointing out how the same idea is represented in two different representations.	If you cut the pieces from Ana's strategy, you can see the 9 pieces from Cal's strategy. They are equivalent.

ing the traditional algorithm, the split method, and compensation (Van de Walle, John, Karp, & Bay-Williams, 2015). To guide the activity used in the middle of the course, PSTs were asked to solve the problem six times three in three different ways: grouping using manipulatives, an array, and using a number line. The last activity introduced an equal sharing problem that stated, "At a restaurant, the waiter brings six sub sandwiches for nine children to share so everyone gets the same amount. How much will each child have?" Three student strategies were presented, each with a distinct drawing showing how the sandwiches could be shared equally among the nine children. After each activity was completed, PSTs were given a writing assessment with our standard questions:

1. What is the big mathematical idea you would like your students to understand?
2. What connections would you address that would highlight the big idea?

Results

To address the first research question examining the types of knowledge connections made by PSTs, we looked at PSTs' responses across the nine assessments used during the course. From our data, we identified three types of connections made by PSTs: superficial knowledge connections, procedural knowledge connections, and conceptual knowledge connections (see Table 1).

Superficial knowledge connections

Connections that use superficial knowledge focus on shallow features of the strategies presented. Within this theme, there are three codes that clarify what we considered a superficial connection. The codes we identified were called: same answer, naming a concept, and identifying various strategies.

Same answer. PSTs identified a connection by describing that multiple strategies led to the same answer. For example, one PST connected students' strategies by saying, "Sandra, Alice, and Milagros used differing strategies and got the same answer." While we believe it is important for PSTs to understand that multiple strategies can produce the same result, we identified this connection as superficial because it does not build upon students' procedural or conceptual understanding.

Naming a concept. PSTs connected strategies by simply naming a mathematical concept used during the solution process. In the quotation below, a PST described the connection as being about *place value*—"the connections that I would address would be place value and how all of the examples kept place value in mind." Although building understanding about place value was the learning goal associated with the mathematics problem, in this instance the PST did not explain how the solution strategies could be used to build understanding about place value, instead the PST only mentioned that place value was evident in all the strategies used. For this reason, we considered a connection such as this one, which names the concept used in the strategies, as a superficial knowledge connection.

Identifying various strategies. PSTs identified that various strategies were used to solve the same problem. In this type of connection, PSTs did not find a relationship between the strategies; they merely mentioned that various strategies were used. For example, "In Alice and Milagros problem, they are both subtracting. However, Milagros does the problem very traditionally whereas Alice breaks her problem down." Here, the PST noticed the problem was solved using different strategies, but the fact that there was no attempt to make a connection between these strategies led us to define this as a superficial knowledge connection. Thus, focusing on superficial knowledge connections will rarely bring forth important discussions about mathematical concepts.

Procedural knowledge connections

The second type of connections we identified were procedural knowledge connections. In this type of connection, PSTs focused their attention on mathematical procedures. We identified two codes for procedural connections, describing a step-by-step solution path and stating the operation that was used to solve the problem.

Step-by-step solution. One PST described the following connection: "strategy 1 got 6/9 and strategy 2 got 2/3. When you simplify strategy 1, you get 2/3 because 6 and 9 are divisible by 3 so 6/3 = 2 and 9/3 = 3 so you are left with 2/3." The PST made a connection between strategies by emphasizing a procedure, simplifying fractions by dividing both parts of the fraction by the same number. We identified this as a procedural knowledge connection because it focused on a step-by-step process for solving the problem. While the connection shown above does help students think about a procedure for simplifying fractions, it falls short of helping students build understanding about why the procedure works.

Stating the operation. PSTs also made connections by stating operations that were used to solve a problem. For example, to connect all three strategies in one activity, one PST said, "Sandra's strategy is drawing blocks, Alice's strategy is grouping and subtracting, and Milagros' strategy is traditional vertical subtracting." Here, a connection was made by identifying the different procedures used in each strategy to illustrate subtraction. However, the PST simply described the different approaches of subtraction and missed an opportunity to connect student strategies by highlighting the use of place value and the properties of operations needed to subtract (CCSSO, 2010).

Conceptual knowledge connections

The last type of connection we identified involved making conceptual knowledge connections. In this theme, we identified the following codes to help demonstrate what we mean by conceptual knowledge connections: describing the big idea, knowing why a procedure works, and using multiple representations. In each of these codes, PSTs described a connection in such a way we believed the connection would assist students in building conceptual understanding of mathematics.

Describing the big idea. The code describing the big idea refers to examples where the PST created a connection that focused on students' understanding of general mathematical rules, facts, and/or definitions. We coded the following connection as conceptual and noted that it is an example of describing the big idea, "I would have them shade the number of 9's in Ana's answer on the 1st sandwich and shade the number of 9's from Ben's answer on the second sandwich to show that 2/3 and 6/9 are equivalent. I would have the student point out the six sets of 1/9 from Cal's strategy in Ben's answer to show how it is also equivalent to 2/3." This PST demonstrated an understanding of the big mathematical idea of equivalency through her assertion that 2/3 and 6/9 are both accurate answers. More specifically, this PST used students' visual fraction models to highlight fraction equivalence, where two fractions with different number and size of parts have the same numerical value or are on the same point on a number line (CCSSO, 2010).

Knowing why a procedure works. An example of this was when a PST made the following connection, "Ben added 1/9, six times to give him 6/9 and Cal represented 1/9 in his drawing and then multiplied it by six to get 6/9. It can be said that Cal and Ben used the same method because although Cal multiplied, multiplication is a form of repeated addition, which is what Ben used." In this case, the PST made a connection by describing the process of multiplication and relating it to repeated addition. This type of conceptual knowledge involves relating concepts to specific procedures and showing understanding for why certain procedures work for particular problems (Crooks & Alibabi, 2014).

Using multiple representations. The final type of conceptual knowledge connection involved multiple representations and was shown when PSTs pointed out the same idea in two different representations. An example of this type of connection was, "I would have the students point out the 6 sets of 1/9 from Cal's strategy in Ben's answer to show how it is also equivalent to 2/3." In a picture included with the response, this PST showed how the six, one-ninths in Cal's strategy could fit within the six, one-ninths in Ben's strategy and then shaded two columns in Ben's strategy to demonstrate how this represented two-thirds. Thus, this conceptual knowledge connection showed an understanding of equivalency by comparing two distinct representations.

Furthermore, we also noticed PSTs sometimes made more than one conceptual knowledge connection within the same response. One PST observed, "the big idea is that *6 x 3* and *3 x 6* is the same thing (commutative property). With all the strategies it can be written in both ways and once they're solved it's the same thing. For example: ||| ||| ||| ||| ||| |||=|||||| |||||| ||||||." Here the PST started by describing the big idea and continued by explaining multiple representations of the same concept. Then, the PST simplified her explanation by using sticks to draw six groups of three and equating them to three groups of six. This demonstrates an understanding of the commutative property in relation to the connection made between the strategies.

As these three types of connections demonstrate, conceptual knowledge connections address ideas that go beyond specific procedures for solving tasks. By addressing the concepts underlying the mathematical strategies used to solve the problem, conceptual connections have the ability to extend mathematical conversations and assist in the development of important mathematical ideas.

Next, we turn our attention to the second research question: How did the types of connections made by PSTs change throughout the study? To investigate this question, we used results from the three assessments that took place during the beginning, middle, and end of the semester. The purpose of this analysis was not to look at changes in individual PSTs, but to look at the sample as a whole to determine which types of connections were most common during which stage of the course. This is helpful in understanding how PSTs' connections changed from the beginning to the end of the course and provides us with valuable information about whether our instructional intervention led PSTs to make more conceptual connections as the course progressed.

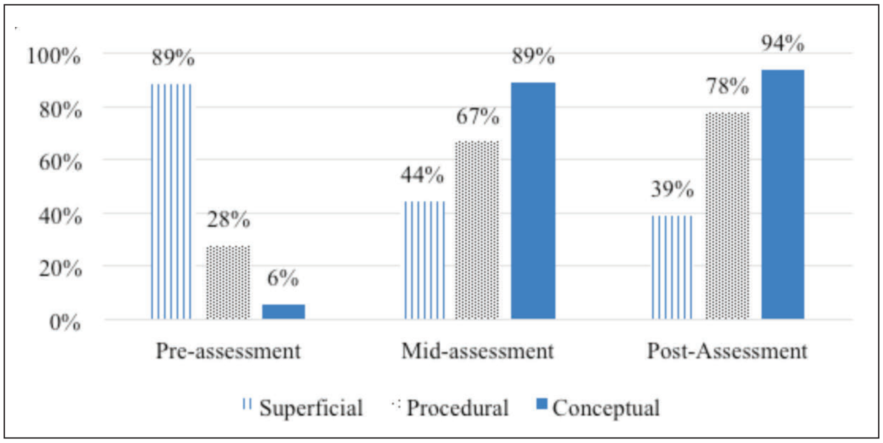


Figure 1. Proportion of students who made each type of connection across activities.

In Figure 2, we can see the proportion of PSTs who made each type of connection by assessment. For example, the 89% above column one on the pre-assessment means that 89% of PSTs in the sample made at least one superficial knowledge connection on the pre-assessment. Preservice teachers were able to make multiple connections on the same assessment. Thus, a PST who made a superficial knowledge connection may have also made a procedural knowledge connection and/or a conceptual knowledge connection on the same assessment.

Comparing the proportion of superficial knowledge connections across the three activities, we see that the amount of this type of connections decreased throughout the study. At the beginning of the course, 89% of PSTs made superficial knowledge connections, while at the end of the course only 39% made superficial knowledge connections. The high proportion of PSTs making superficial knowledge connections on the first activity was coupled with a relatively low proportion of PSTs making either procedural or conceptual knowledge connections. This suggests that PSTs' knowledge at the beginning of the course was focused on superficial types of connections. The results at the end of the course looked very different. The last activity showed that PSTs' knowledge about connections focused less on superficial knowledge and more on procedural and conceptual knowledge connections. In fact, by the end of the course, 94% of PSTs identified at least one conceptual knowledge connection, up from 6% at the beginning of the course.

When examining the connections made on the last assessment, we noticed PSTs were understanding how to use the big idea and the strategies together to build conceptual understanding. For example, one PST said, "I would also explain to them how 6/9 simplifies to 2/3 so they can understand that both fractions are still equivalent. I would even take one of the drawings from Ana's picture and one from Ben's and shade in 2/3 and 6/9 so they can visually see they're equal." Here, we see how this student first makes a connection when she describes the big idea of equivalence using the strategies provided, and then, she makes another connection as she continues to clarify the big idea by using its definition to demonstrate with multiple representations how two fractions are equivalent.

Discussion and Implications for Further Research

When considering the ability of a connection to promote mathematical understanding, focusing exclusively on superficial knowledge connections may provide students with opportunities to see obvious similarities between strategies, such as having the same answer, but it also hinders the direction of the whole classroom discussion when thinking about big mathematical ideas. Procedural knowledge connections are a salient part of a classroom discussion because they help students develop instrumental understanding; however, these connections do not encourage students to move beyond describing a procedure to understand why it works. Conceptual knowledge connections, on the other hand, promote rich discussions that lead to the big idea of the lesson and makes the mathematical learning goal visible to the entire class. Conceptual connections build relational understanding as teachers work to help students generalize ideas and identify them across multiple mathematical representations.

Ensuring PSTs are able to make connections and understand the roles connections make in learning mathematics is necessary, but PSTs must also learn how to draw connections out of their students. Throughout the course, we modeled the behavior of a teacher who is drawing out connections from his or her students; however, we never explicitly worked with PSTs to help them acquire this skill. We consider this a shortcoming in our intervention and plan on including this in the next iteration of our teaching. As a starting point, we note several questions that PSTs could be instructed to use when working with their own students that we believe assist in getting students to think about connections. These are:

- What makes you say that?
- What are some similarities (or differences) you see between the solution strategies?
- How do these similarities and differences help you understand the big idea?
- How are you able to see X represented in solution Y?
- How does X representation help you understand why procedure Y works?
- We also note examples of specific questions we asked during the activities that assisted PSTs in making mathematical connections. For example, when discussing the pre-assessment, after PSTs had a chance to write about the connections they saw, we used the following questions to push their thinking further:
- Where do you see the use of place value in strategy one? Where do you see place value in strategy 2? How would you use this to help your students understand the big idea?
- In this strategy the student is borrowing from the tens place to subtract. Is this process happening in another strategy?

When discussing the activity from the middle of the semester, we inquired:

- Where do you see the row of six represented in any of the other strategies?
- How does the array show the addition, 6 + 6 + 6?

Finally, when discussing the last activity, we asked:

- Are 2/3 and 6/9 equivalent? Use the strategies to defend your answer.
- How can we use the strategies to show fraction equivalence?

Subsequently, we posit that being transparent about the questions we use in class and why will be helpful in the future.

As educators, this action research project was a practical way to improve our practice, while simultaneously creating knowledge that is valuable to other educators (Somekh & Zeichner, 2009). Because we had a specific goal in mind, that of making connections, we were able to concentrate on a crucial piece of developing understanding in mathematics. We also saw the value in this type of research and were eager to share it with other educators. This is one of the many purposes of action research, sharing useful information with peers in similar positions (Somekh & Zeichner, 2009). Although this study was conducted with elementary PSTs, research shows secondary PSTs compartmentalize mathematical ideas and have difficulty making connections between solution strategies (Even, 1993; Moon, Brenner, Jacob & Okamoto, 2013). It would be interesting to implement this intervention in a secondary mathematics content and methods course to see if similar types of connections emerge. Hence, we understand that more iterations of this action research project need to happen to further understand the types of connections PSTs make and how they improve over time. Nevertheless, this study adds to the body of literature on innovative practices in mathematics education and in action research.

Appendix

Pre-Assessment

You present your second grade class with the problem 70-59. Your students solve it in the following ways.

Traditional Algorithm	*Compensation Method*	*Split Method*
60	70 – 59	70 – 59 =
70	70 – 60 = 10	70 – 50 – 9 =
–79	+1	20 – 9 = 11
11	11	

a. What big mathematical idea can you address by using this problem?

b. What connections can you make between the strategies to highlight the big idea?

Mid-Assessment

Students present three ways to solve the problem 6 x 3: grouping using manipulatives, array, and number line. Solve 6 x 3 using these three strategies, then use these strategies to answer the following questions.

a. What big mathematical idea can you address by using this problem?

b. What connections can you make between the strategies to highlight the big idea?

Post-Assessment

Imagine you are a 3rd grade teacher. In your class, you are teaching equal sharing using the following problem:

At a restaurant, the waiter brings 6 sub sandwiches for 9 children to share so everyone gets the same amount. How much will each child have?

Three of your students' strategies are below. Use these strategies to answer the following questions.

a. What big mathematical idea can you address by using this problem?

b. What connections can you make between the strategies to highlight the big idea?

Ana's Strategy:

1	2	3	4	5	6	7	8	9
1	2	3	4	5	6	7	8	9

Each number represents a child.
Each child gets 1/3 + 1/3 = 2/3 of a sub sandwich.

Bens's strategy:

1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

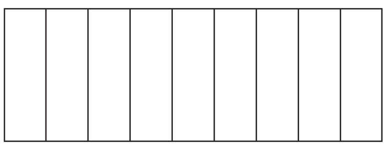
1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

1	2	3
4	5	6
7	8	9

1/9 + 1/9 + 1/9 + 1/9 + 1/9 + 1/9 = 6/9
6/9 of a sub each.

Cal's strategy:



Each sub split into 1/9 pieces and each child will get 1/9 of each sub.
There are 6 subs so each child will get 1/9 x 6 = 6/9

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