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A Century of Leadership in Mathematics and Its Teaching

Examining the Classroom Experience

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Selecting Tools to Model Integer and Binomial Multiplication

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ABSTRACT Mathematics teachers frequently provide concrete manipulatives to students during instruction; however, the rationale for using certain manipulatives in conjunction with concepts may not be explored. This article focuses on area models that are currently used in classrooms to provide concrete examples of integer and binomial multiplication. The innovation of combining the representations for negative numbers with both algebra tiles and *Algeblocks* is provided, with a mathematical justification for its development. Teachers' effective integration of tools such as these in mathematics instruction can help students develop conceptual understanding and procedural fluency.

KEYWORDS *integer multiplication, binomial multiplication, area model, algebra, negative integer multiplication*

In mathematics education, concrete manipulatives and visual materials are often provided to aid students' exploration and understanding of mathematical concepts. Though hands-on models have been espoused for almost a century (see Sowell, 1989 for a historical synopsis), research related to teachers making informed decisions about which ones to select is still limited. Ertle (2006) demonstrates the importance of bringing this awareness to light; more specifically, the need for knowledge of how, when, and why to use manipulatives. Textbooks and curricular materials often include concrete manipulatives and visual materials, and these resources include the "how" and "when." Unfortunately, descriptions of "why" they should be used, in a mathematical sense, are absent. In this article, the innovation of combining the representations for negative numbers with both algebra tiles and *Algeblocks* is provided, with a mathematical justification for its development.

In *Principles to Action* (National Council of Teachers of Mathematics (NCTM), 2014), NCTM sets forward six guiding principles for school mathematics. This article targets the principle of Tools and Technology, described in this way: "An excellent mathematics program integrates the use of mathematical tools and technology as essential resources to help students learn and make sense of mathematical ideas, reason mathematically, and communicate their mathematical thinking" (p. 78). We outline area models that are currently used during instruction as tools to provide concrete examples of integer and binomial multiplication. Then, we posit an emergent model and a mathematical justification for why we have developed it.

The purpose of this article is to provide an essential resource for integer and binomial multiplication that is—drawing from Doll's (1993) four R's of rigor, richness, relations, and recursion—mathematically rigorous, conversationally rich, filled with relations among mathematical concepts, and can move recursively from whole numbers to integers to binomials. Teachers can use the tools to facilitate complex conversations (Pratt, 2008) where the teacher listen for differences (Davis, 1997) to draw out multiple perspectives and representations, and enact effective teaching practices (NCTM, 2014). First, we provide an overview of area models used to explore multiplication in mathematics education, specifically focusing on base-10 blocks, algebra tiles, and *Algeblocks*. We then problematize the overgeneralization of algebra tiles when expanding from the set of whole numbers to the set of integers. Next, a resolution of this problem is offered, with a demonstration of how to make the expansion more coherent across the set of integers. Finally, we conclude with a series of activities that can be adapted by mathematics educators to model this resolution and utilize area models in a mathematically rich manner.

Area Models

Area models generate opportunities for students to make rich connections between operations on numbers and their algebraic representations (Richardson, Pratt, & Kurtts, 2010; Pratt, Richardson, & Kurtts, 2011). Area models can be used in conjunction with the concepts of multiplication of whole numbers, integers, and polynomials (Wheatley, 1998). The area models of the products (11) (12) and $(x+1)$ $(x+2)$, as described in Richardson, Pratt, and Kurtts (2010), reveal structural similarities (See Figure 1). Base-10 blocks are used to represent 11 as (10 + 1) and 12 as (10 + 2), while *Algeblocks* are used to model $(x+1)$ $(x+2)$. By placing the structures of these two products as areas side-by-side, conversations around the meanings of base-10 to base-x can be explicitly guided to facilitate understandings of the place values associated with the products of $10^2 + 2(10) + 1(10) + 2$ and $x^2 + 2x + x + 2$, respectively. This is an example of what Jacobs, Franke, Carpenter, Levi, and Battey (2007) would consider a task that elicits algebraic reasoning as generalized arithmetic.

In the right diagram of Figure 1, teachers could elect to use algebra tiles in the place of *Algeblocks*, and the structural similarities between (11) (12) and $(x+1)$ $(x+2)$ would still be evident. There is minimal difference in the representation of algebra tiles and *Algeblocks* when modeling operations involving the set of whole numbers. However, when the values shift to integers, representations of negative values create a significant difference in the models. This difference is important for those who are teaching integer and binomial multiplication while modeling the operations using manipulatives. The delineation between algebra tiles and *Algeblocks* is an important conversation in which teachers should engage to provide more in-depth understandings of the mathematical concepts these tools represent. We agree with Ertle (2006) that this awareness can enhance teachers' pedagogical content knowledge.

Algebra Tiles

Algebra tiles have become the most prolific set of concrete manipulatives with which to explore integer and binomial multiplication. Bruner (1965) used them in his interviews with eight year olds, when they were made from pieces of wood. The shapes of the tiles displayed different areas. Since then, they have been commercially reproduced by different manufacturers, and more recently developed into virtual models as apps (e.g., *Algebra Tiles,* HMH FUSE Algebra 1) and applets (e.g., NCTM Illuminations, National Library of Virtual Manipula-

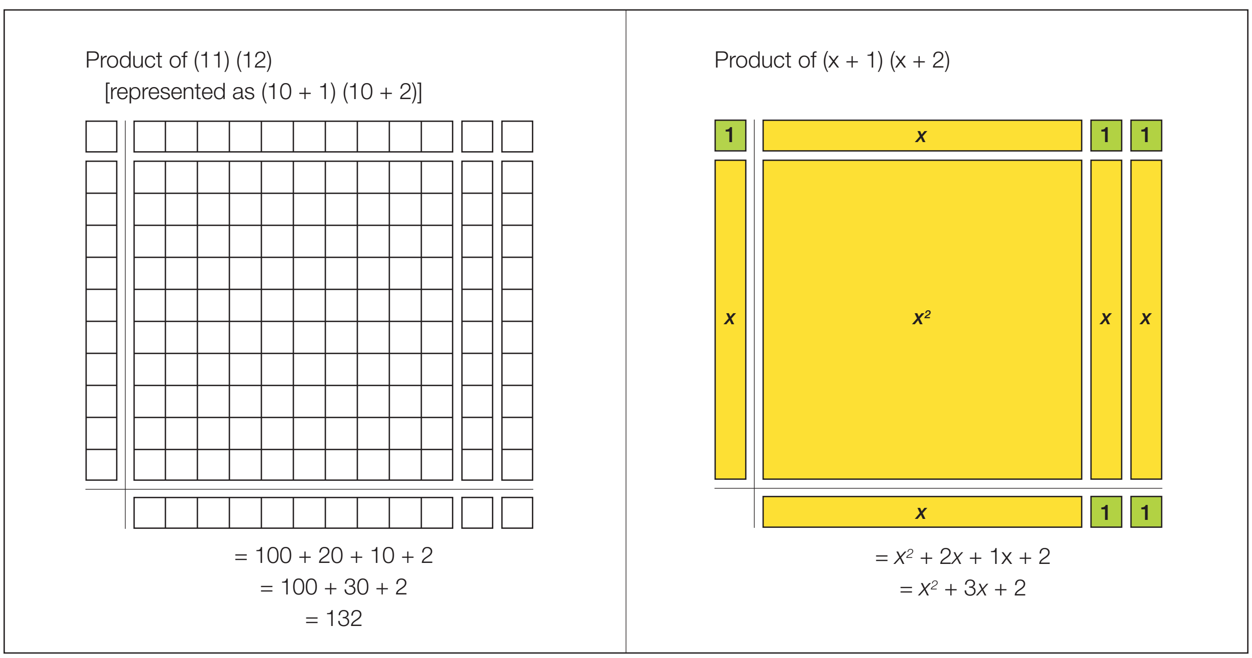

Figure 1. A comparison of the models of (11) (12), using base-10 blocks, to (x + 1) (x + 2), using *Algeblocks.*

tives). The original algebra tiles were created by Bruner (1965) to assist young learners with constructing area models involving operations on whole numbers and algebraic expressions within the set of whole numbers. Algebra tiles were then expanded to include the set of integers. Now, the tiles are two-sided, with each side showing a different color. On one side, the color represents a positive area and is varied (usually blue, green, or yellow, depending on the manufacturer or technical developer) based on the dimensions of the piece. Consistently across all models is the other side, which is red; this represents a negative area—no matter the dimensions of the piece—and signifies that the piece's area is the opposite of its positive value.

Modeling addition and subtraction with algebra tiles requires setting the tiles on a surface with the color facing up to represent positive or negative values, and then collecting like terms. Positive pieces are placed together; negative pieces are also placed together. Any like terms that are opposites (additive inverses) are lined up then "zeroed out" (commonly referred to as zero pairs). The resulting collection is summed to find the solution. For example, Figure 2 models –4 + 2 using algebra tiles.

For multiplication, algebra tiles are oriented to show the area of a rectangle using a singular base times height mat. The orientation of how the tiles are placed to show area varies based on the manufacturer. All models determine rectangular area using base times height, but some show the base across the top while others show base across the bottom. The height is consistently oriented to the left. For example, Figure 3 shows the operation of (–2) (+3) with the height of –2 on the left and base as +3 across the bottom. The resulting product is –6 because of the rule that a positive number times a negative number results in a negative number. Alternatively, the height could be +3 on the left and the base as –2 across the bottom, and the result would be the same.

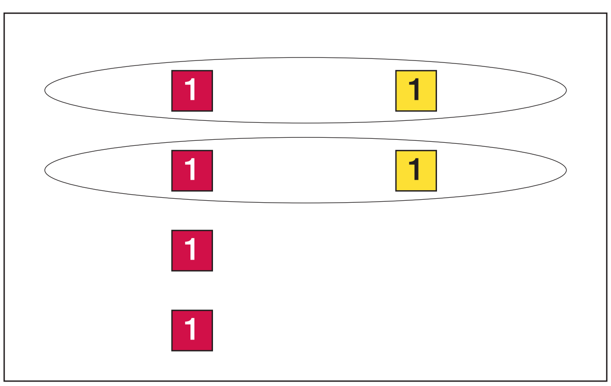

Figure 2. Algebra tiles to model the integer addition of –4 + 2, showing the resulting sum of –2.

Due to the way algebra tiles are placed on this singular base times height mat, there are limitations regarding students' conceptual understanding of the mathematical reason for flipping pieces over. As modeled in Figure 3, the rule for multiplying a negative number by a positive number is what dictates flipping the pieces. Engaging in why they are flipped beyond this rule is often not examined or discussed. Teachers are told to flip them to represent their opposites, and in turn, students are told the same. The how and when are discussed, but not the why.

Algeblocks

In contrast to algebra tiles, the creators of *Algeblocks* approached the representation of negative values differently. The pieces do not have opposite sides displaying different colors. Instead, the pieces are placed on one of three mats (basic mat, sentence mat, and quadrant mat) that accompany the set of blocks to model addition/subtraction, solving equations, or multiplication/division, respectively. The distinction made for positive versus negative values has to do with where the pieces are placed on the mat (there is a "negative side" and a "positive side"), rather than a difference in color.

Mathematically, the mats are concrete representations of number lines. For example, the task of simplifying –4 + 2 can be shown by placing four units on the negative side of the basic mat and two units on the positive side (Figure 4). Additive inverses are placed alongside each other on opposite sides of the mat, and then removed. What remains is two units on the negative side, showing the solution of –2. A discussion of the Inverse Property of Addition can be included during instruction. Though

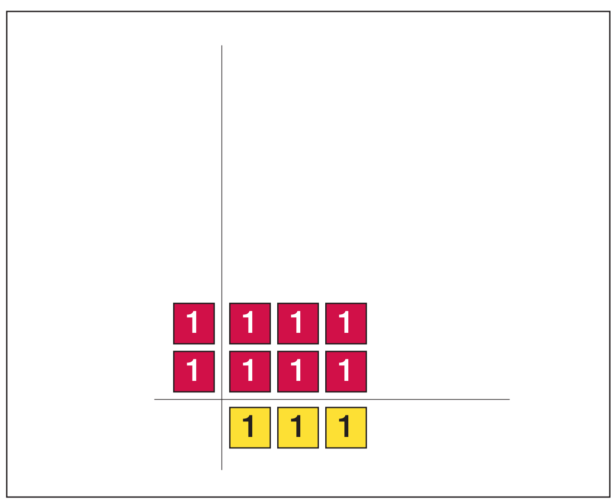

Figure 3. Algebra tiles to model the integer multiplication of (–2) (+3), where the product –6 is represented by the filled in area.

this is similar to the use of algebra tiles, the difference occurs when translating this model to a number line to compare the meanings of positive and negative values.

Moving from the basic mat to the quadrant mat, *Algeblocks* can be used to model multiplication and division of integers in a different way than algebra tiles. The quadrant mat is a representation of the coordinate plane, and the factors are placed on the x-axis and y-axis to show that base times height produces a rectangular solution. While the *Algeblocks* and algebra tiles both conceptualize multiplication using area models, the quadrant mat compared to the singular base times height mat is what sets them apart. For example, using *Algeblocks* and the quadrant mat to model the operation (–2) (+3) as shown in Figure 3, –2 is placed on the x-axis to the left the origin, and +3 is placed on the y-axis above the origin (See Figure 5a). The resulting area is –6 because there are six units placed in the second quadrant (See Figure 5b). (Alternatively, –2 could be placed on the y-axis below the origin and +3 on the x-axis to the right of the origin. Six units would then be placed in the fourth quadrant, for the same answer of –6.)

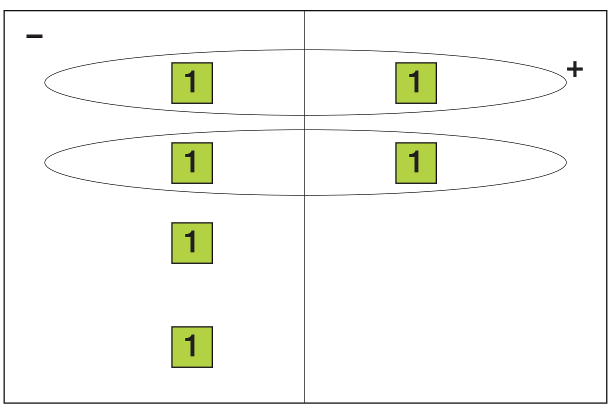

Figure 4. Algeblocks and the basic mat to model the integer addition of –4 + 2, showing the resulting sum of –2.

An additional benefit of the *Algeblocks* kit is the track that is included. The track is placed on the quadrant mat, and the factors are placed inside the track. When the factors are multiplied and the area is filled in, the track can be lifted to see the product only (See Figure 5c.). Then, when the solution is recorded, the factors are not included in the product. When students use the track and lift the factors away, they are able to articulate the correct product. This becomes even more important when binomials are multiplied and the resulting product is placed in all four quadrants.

This elicits a strong mathematical rationale for why the product is negative, including directional values. Further, there are opportunities to facilitate conversations about why a positive number times a negative number results in a negative number, as well as the notion that on the coordinate plane, this shows a negative relationship. Functions and slope of linear functions can be discussed to facilitate connections between these concepts.

An Emergent Model

As colleagues and researchers, we bring with us different choices for instruction with secondary students as well as pre-service and in-service mathematics teachers, one using algebra tiles and the other using *Algeblocks*. When we came together to select instructional materials for a professional development grant, we debated regarding which set of manipulatives to purchase. Initially, we selected *Algeblocks* kits. As we presented workshops about area models for integer and binomial multiplication, we continued our debate regarding the tools. We came to realize that by integrating the tools together, the act of flipping pieces over provides a positional representation for the meaning of multiplicatively inversely-related quantities (Pratt & Eddy, 2016).

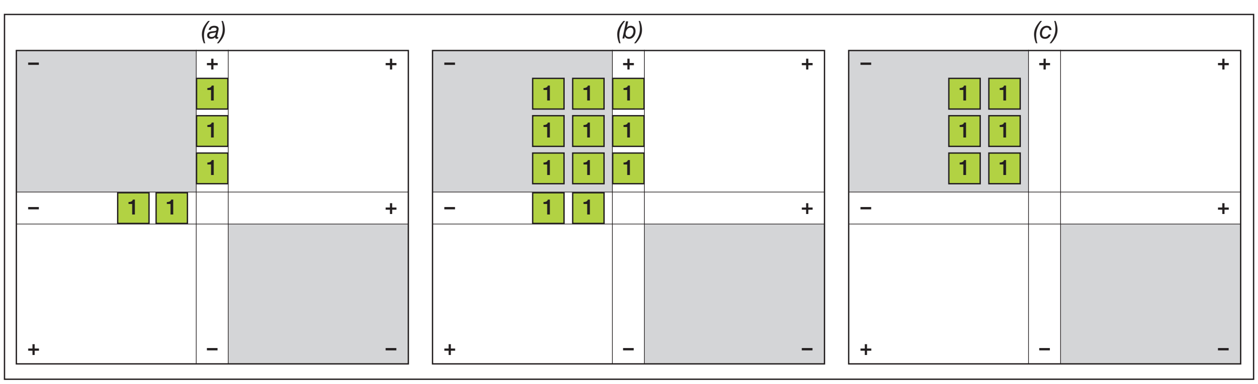

Figure 5. (a) Algeblocks on the quadrant mat to model (–2) (+3); *(b)* completing the area by dimensions; and, *(c)* the resulting area of –6 after lifting the factor track.

What we argue with our innovation is that the soundest mathematical representation of integer and binomial multiplication arises when the algebra tiles are placed on the quadrant mat included in the *Algeblocks* kit. For example, suppose the area of +3 is located in Quadrant I (See Figure 6a). To represent the area of –3, we can reflect the Quadrant I tiles either across the *y*-axis to be located in Quadrant II (Figure 6b), or across the *x*-axis to be located in Quadrant IV (Figure 6c). Either way, under a reflection, the color of the set of three algebra units is changed, while the area remains the same.

The geometric transformation of a reflection is a mathematical justification for why the tiles should be flipped. Figure 5 above shows the operation of (–2) (+3) using *Algeblocks*. If this expression was modeled using algebra tiles on the quadrant mat instead, the base would be reflected across the *y*-axis to transform its color from +2 (green) to –2 (red). The measurement of the base of two negative units (to the left) and height of three positive units (up) results in a negative area because it is in Quadrant II (See Figure 7). Next, a discussion comparing the result of (+2) (+3) that would place the area in Quadrant I could lead to opportunities for reasoning mathematically.

Traditionally, algebra tiles are modeled using only one quadrant, whereas the *Algeblocks* quadrant mat shows the coordinate plane with all four quadrants. In previous research conducted (Pratt, 2017), a participant questioned why Quadrants II and IV on the quadrant mat were shaded (as provided by the manufacturers). She did not want to take that information for granted, nor just be told they are negative. The incorporation of the algebra tiles with the quadrant mat provides an excellent opportunity to engage in the question as to why Quadrants II and IV represent negative values and Quadrants I and III represent positive values.

For example, the +3 used in Figure 6 above illustrates the power of reflection across both the *y*- and *x*-axes (Figures 6b and 6c, respectively). Using the Multiplicative Identity (+1) (+3) = +3, the factors are placed in the track corresponding to the *x*- and *y*-axes (Figure 8a) that formed the area. Now, when the +3 is reflected across the *y*-axis for a result of –3 (as shown above in Figure 6b), the factors in the track to represent this area are –1 on the *x*-axis and +3 on the *y*-axis (below in Figure 8b). This represents the product (–1) (+3) = –3. This geometrically proves that a negative number times a positive number

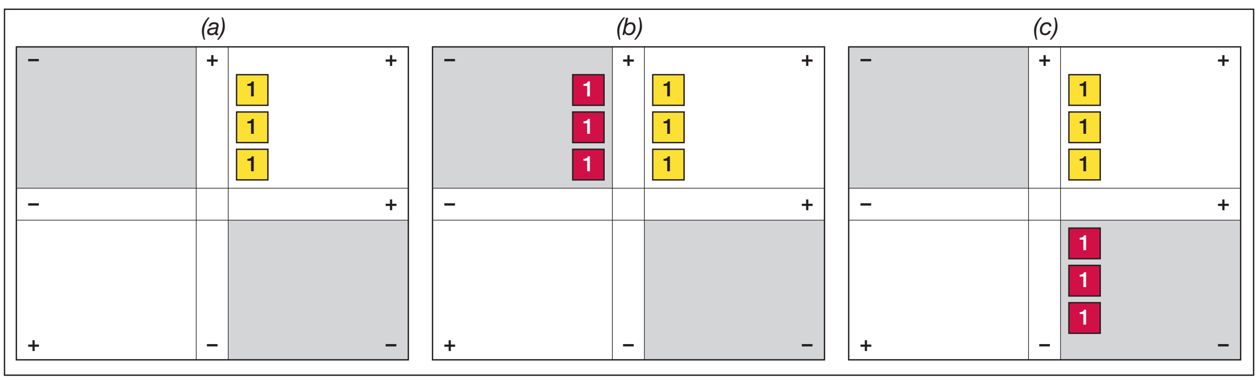

Figure 6. (a) Algebra tiles on the quadrant mat beginning with the area of +3; *(b)* reflected across the *y*-axis in Quadrant II to show –3; and, *(c)* reflected across the *x*-axis in Quadrant IV to show –3.

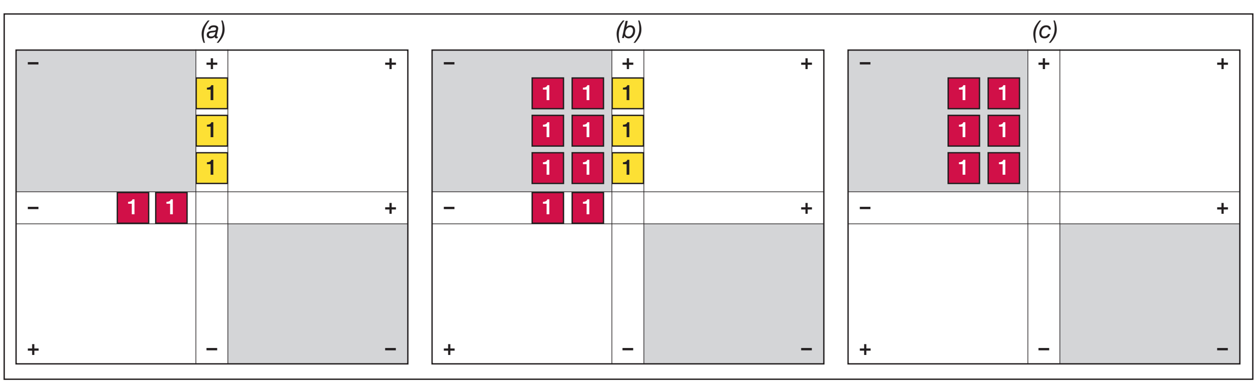

Figure 7. (a) Algebra tiles on the quadrant mat to model (–2) (+3); *(b)* completing the area by dimensions; and, *(c)* the resulting area after lifting the factor track of –6.

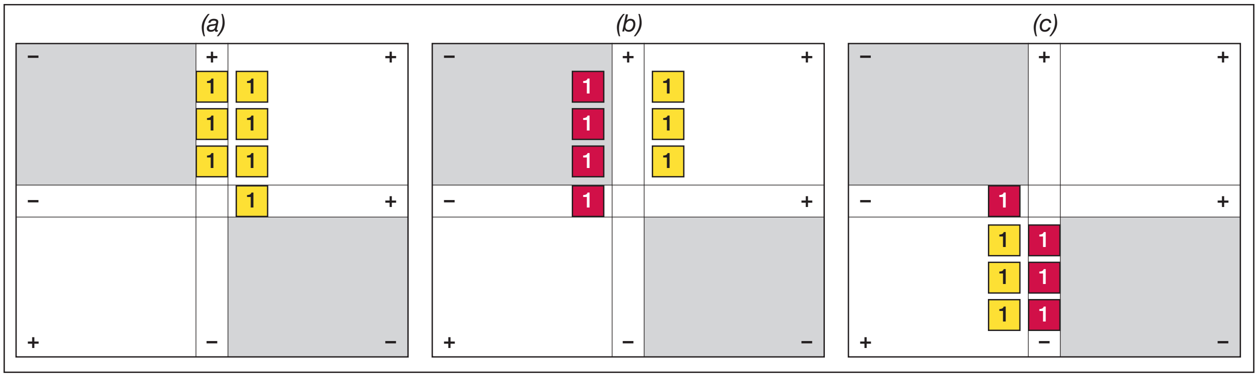

Figure 8. (a) Algebra tiles on the quadrant mat to model (+1) (+3) = +3 in Quadrant I; *(b)* reflecting +3 across the *y*-axis to model (–1) (+3) = –3 in Quadrant II; and, *(c)* reflecting –3 in Quadrant II across the *x*-axis to model (–1) (–3) = +3 in Quadrant III.

equals a negative number. Then, a reflection of –3 in Quadrant II across the *x*-axis results in +3, and the factors in the track are –1 on the *x*-axis and –3 on the *y*-axis (Figure 8c). This represents the product (–1) (–3) = +3, thus geometrically proving that a negative number times a negative number equals a positive number. It can now be seen that beginning with (+1) (+3) = +3 (Figure 8a) and reflecting across both the *x*- and *y*-axes, the result is (–1) (–3) = +3 (Figure 8c). This series models why Quadrants II and IV on the quadrant mat represent negative values, while Quadrants I and III represent positive values.

We have found that flipping the algebra tiles' colors as the area is reflected into an adjacent quadrant—combining the representations for negative numbers employed by both algebra tiles and *Algeblocks*—stimulates rich dialogue among pre- and in-service teachers, leading to their deeper conceptual understanding. The connections between this and other algebraic concepts, such as linear functions, abound.

Binomial Multiplication

In our research with prospective middle grades mathematics teachers (Pratt, Richardson, & Kurtts, 2011; Pratt, 2017), we have found that many fail to acquire a relational understanding (Skemp, 1978) of the concept of binomials. While algorithms involving binomials can be performed with accuracy during instruction (instrumental understanding), the depth of the meaning of "binomial" is often lacking, and real-world connections are often not present (relational understanding). Because of this, we developed a series of tasks that compare the structure of integers represented by base-10 blocks to binomials represented by algebra tiles on the quadrant mat. In Figures 9 and 10, we demonstrate this using (7) (11) represented as (10 – 3) (10 + 1), and compare it to $(x - 3)(x + 1)$.

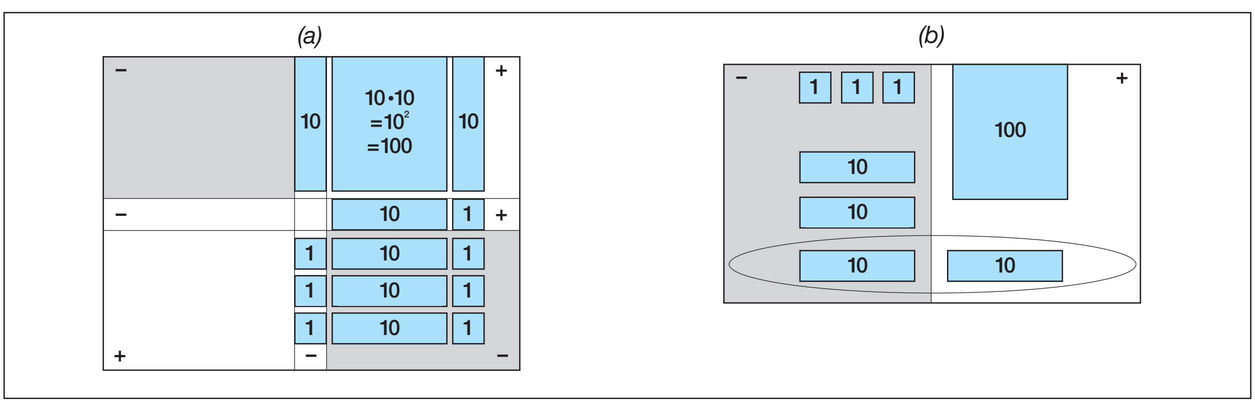

Figure 9. (a) Base-10 blocks on the quadrant mat to model (7) (11) as (10 – 3) (10 + 1), resulting in 100 + 10 – 30 – 3 = 77; and, *(b)* with the product on the basic mat to show the additive inverse, 100 + [+ 10 – (10)] – 20 – 3 = 100 – 20 – 3 = 77.

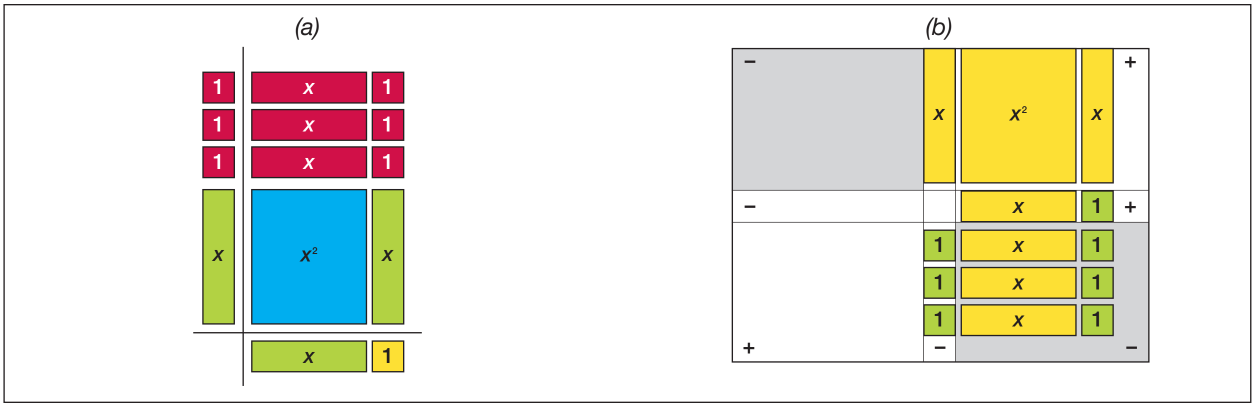

Figure 10. (a) Algebra tiles and *(b) Algeblocks* to model (x – 3) (x + 1).

Integer multiplication

For example, we provide the prompt of showing the product (7) (11) with the base-10 blocks, including the challenge to be as efficient as possible in generating the product. With some prompting and encouragement of thinking, the model of (10 – 3) (10 + 1) is created to show an area model for (7) (11). The blocks can be placed on the basic mat beside the quadrant mat to show that the mathematical process is 100 + [10 – 10] – 20 – 3 = 100 – 20 – 3 = 77 (See Figure 9).

Binomial multiplication in the set of integers

Following this example, the prompt is given to find the product (x – 3) (x + 1) (See Figure 10). These binomials are intentionally selected to facilitate a comparison to (7) (11) (Figure 9). Figure 10 shows the resulting area with algebra tiles using the traditional methods and contrasts this with the resulting area using *Algeblocks* and the quadrant mat.

Our emergent model is shown in Figure 11 where the same task is displayed, but algebra tiles are placed on the quadrant mat. While Figure 10 shows the area is the same value, Figure 11 builds from students' prior knowledge of the coordinate plane and assists students' understanding of multiplying two binomials and the reason that part of the trinomial has negative values. These connections between algebra tiles and the quadrant mat from *Algeblocks* strengthen conceptual understandings of integer and binomial multiplication (Pratt & Eddy, 2016).

In Figure 12, the area, $x^2 + x - 3x - 3$, is placed on the basic mat to engage in discussions about collecting like terms, showing that by the Additive Inverse Property, the units of size x will combine. The mathematical process is $x^2 + [x - x] - 2x - 3 = x^2 - 2x - 3$. The side-by-side comparison of the quadrant mat and the basic mat shows that although the area resulted in four separate terms, two of those terms can be combined and the final product is a trinomial. This intentional step in the process will assist students when they learn to factor trinomials, providing them the opportunity to recognize that the dividend will be a rectangle whose area can be represented in multiple ways.

A second task to model binomial multiplication is

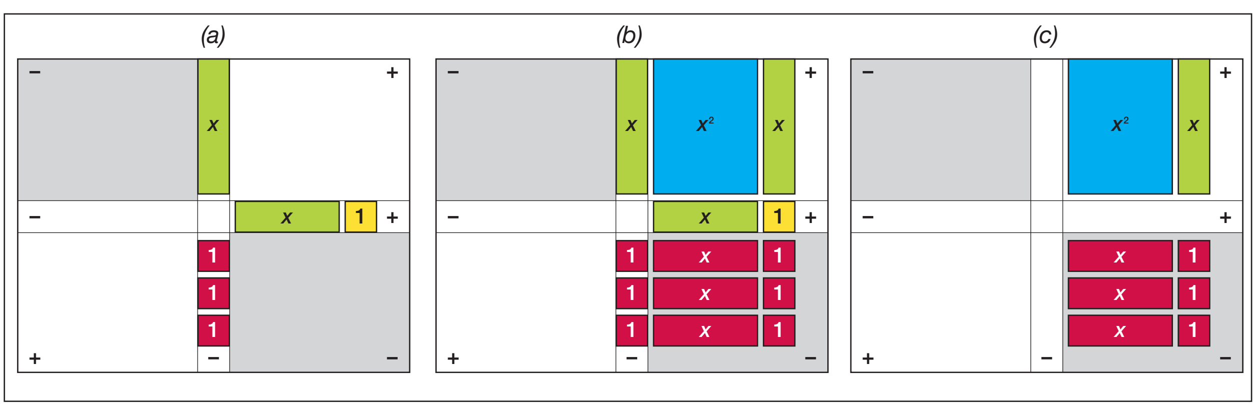

Figure 11. (a) Algebra tiles on the quadrant mat to model (x – 3) (x + 1); (b) completing the area by dimensions; and, (c) the resulting area after lifting the factor track (x2 + x – 3x – 3).

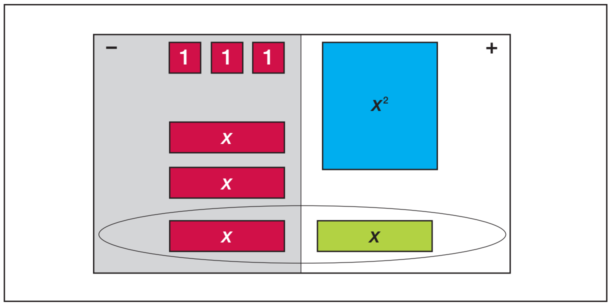

Figure 12. The product of the algebra tiles placed on the basic mat to show the additive inverse, $x^2 + [+ x - (x)] - 2x - 3 = x^2 - 2x - 3$.

presented in Figure 13, where two binomials are placed across both the x- and the y-axes, generating an area with terms located in all four quadrants. The algebra tiles on the quadrant mat model $(x - 2)\ (x - 1)$. In this example, one of the factors includes placing x above the origin and -2 below the origin. The other factor shows x placed to the right of the origin and -1 to the left of the origin. When each dimension is multiplied, the resulting area shows x^2 in Quadrant I, $-x$ in Quadrant II, $-2x$ in Quadrant IV, and $+2$ in Quadrant III. When the product is placed on the basic mat to collect like terms (see Figure 14), the x^2 and the $+2$ are placed in the positive region, while both $-x$ and $-2x$ are placed in the negative region. There are no additive inverses, and the like terms are combined to show that the product includes $-3x$. The mathematical process is $x^2 - x - 2x + 2 = x^2 - 3x + 2$.

Discussion

Koellner and colleagues (2007) argue that "the improvement of students' opportunities to learn mathematics depends fundamentally on teachers' skill and knowledge" (p. 273). As teacher educators, our work supports the notion that pre- and in-service mathematics teachers need to engage in the reasons why certain manipulatives are effective models of mathematical concepts if they are to make quality decisions to "carefully select which representations to use, as student learning is influenced by the representations to which they are exposed and some representations reinforce misconceptions" (Mitchell, Charalambous, & Hill, 2014, p. 42). The NCTM (2014)

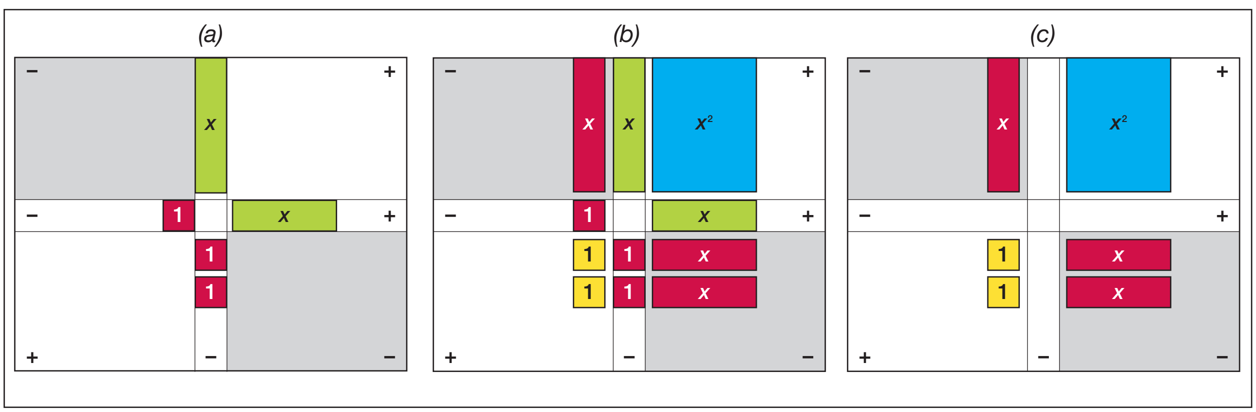

Figure 13. (a) Algebra tiles on the quadrant mat to model $(x - 2)\ (x - 1)$; *(b)* completing the area by dimensions; and, *(c)* the resulting area after lifting the factor track $(x^2 - x - 2x + 2)$.

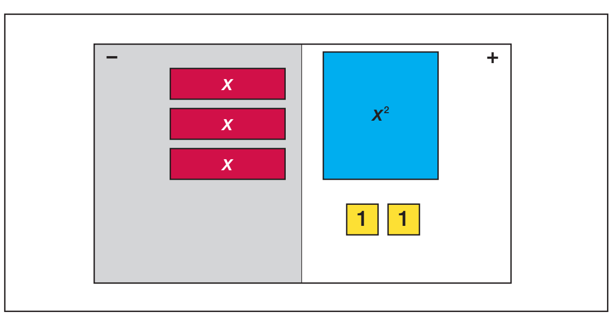

Figure 14. The product of the algebra tiles placed on the basic mat to show collecting like terms, $x^2 - x - 2x + 2 = x^2 - 3x + 2$.

guiding principle of Tools and Technology is an integral part of effective instruction, when used appropriately. The use of algebra tiles on the *Algeblocks* quadrant mat affords the opportunity for teachers to implement the mathematical teaching practices listed in NCTM (2014):

> Establish mathematics goals to focus learning; implement tasks that promote reasoning and problem solving; use and connect mathematical representations; facilitate meaningful mathematical discourse; pose purposeful questions; build procedural fluency from conceptual understanding; support productive struggle in learning mathematics; and, elicit and use evidence of student thinking. (p. 10)

Concrete manipulatives can be used to demonstrate understandings as well as misconceptions, even when those cannot be verbally articulated. By using algebra tiles on the *Algeblocks* quadrant mat for integer and binomial multiplication, teachers can build students' conceptual understanding before moving to procedural fluency, which is one of the effective teaching practices espoused by NCTM (2014).

When algebra tiles are used with the *Algeblocks* mats in conjunction with the teaching practices that initiate conceptual understanding instead of mere procedural algorithms—telling others what to do (Lobato, Clarke, & Ellis, 2005)—instructors will be able to "open doors for many students who struggle with abstract symbols" (Moyer & Jones, 2004, p. 29). Drawing on inquiry-based instruction that incorporates tools, we argue that teachers who use our proposed innovation can generate complex conversations (Pratt, 2008) that are mathematically rigorous, conversationally rich, filled with relations among mathematical concepts, and can move recursively from whole numbers to integers to binomials. We strongly encourage educators to use algebra tiles together with the *Algeblocks* quadrant mat to model integer and polynomial multiplication in order to build mathematical understanding and fluency.

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