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## Mathematical Creativity, Cohen Forcing, and Evolving Systems: Elements for a Case Study on Paul Cohen

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The Evolving Systems approach to case studies due initially to Piaget-contemporary Howard Gruber, and complemented by subsequent work on sociocultural factors developed by Mihaly Csikszentmihalyi and others, provides an inroad for examining creative achievements in a variety of domains. This paper provides a proof of concept for how one might begin to explore questions about the creative development of Cohen forcing, a powerful technique in Set Theory and Mathematical Logic.

*Keywords:* Paul Cohen, Continuum Hypothesis, creativity, forcing, Howard Gruber, Set Theory, mathematics, mathematical creativity

### Introduction

During the 1900 International Congress of Mathematicians in Paris, the preeminent German mathematician David Hilbert announced his list of 23 unsolved problems. Within the domain of Mathematics, these problems would be among the most influential over the next century and beyond. Their impact is still felt today; for example, the eighth problem on Hilbert's list, known as the Riemann Hypothesis, remains unsolved as of 2013: this is despite a million dollar (USD) prize announced by the Clay Mathematics Institute in 2000 for a solution (Bombieri, 2000).

Among Hilbert's 23 problems, the first one listed is known as the Continuum Hypothesis (CH) and traces its roots back to work by Georg Cantor, the founder of Set Theory, in 1878. The problem was ultimately tackled in two parts: the first half was completed by Kurt Gödel around 1938, and the second half was disposed of by Paul Cohen in 1963. By the time of his published solution, Gödel was already famous for several results, most notably his Incompleteness Theorems, and he remains widely regarded as one of the greatest logicians to live. For the reader unfamiliar with Gödel's work, Albert Einstein is known for having remarked later in life that his "own work no longer meant much, [and] that he came to the Institute [for Advanced Study at Princeton] merely . . . to have the privilege of walking home with Gödel" (Goldstein & Alexander, 2006). Nevertheless, it was to be the 29-year-old Cohen who ultimately answered in full the Continuum Hypothesis. Though less a household name than Gödel or Einstein, it is Cohen who serves as the focus of this paper. More precisely, Cohen is known for having taken up CH only a year before his published solution appeared, and, despite not being a specialist in either Set Theory or

Mathematical Logic, having successfully resolved the top outstanding problem in both areas with his development of a new technique known as *forcing*.

A brief outline of the remaining sections is as follows: The next section will include general remarks on pure mathematics in the context of case studies in creativity, as well as specific remarks on the relatively minimal mathematical background necessary to read this paper. (For the mathematically inclined reader, the reference list includes upper-level texts on forcing, as well as Cohen's work on CH in particular.) The paper will then trace the evolution of Cohen's development of forcing, with an eye to the creativity literature and a focus on the Evolving Systems approach as described in Gruber & Wallace (1999) and Brower (2003). The subsequent section will examine some of the sociocultural factors underlying Cohen's work, before concluding with a short discussion of the ramifications of this case study for Creativity Studies and Mathematics Education.

### Pure Mathematics in Creative Case Studies

How does one behold pure mathematics? One can behold paintings by gazing upon them, so that even a reader unfamiliar with the specifics of Picasso's *Guernica* has at least an entry point when reading a case study on its creation (Weisberg, 2004). Similar remarks can be made about architecture, where one can look at blueprints or the final structure built (Weisberg, 2011); the hard sciences, where one can look at diffraction images of DNA or a 3D model of the double-helix in a case study on the work of Watson and Crick (Weisberg, 2006); and so forth. Results of pure mathematics are more akin to literary works in a foreign language, but to translate the words is highly nontrivial, as they frequently encode

concepts that are not part of everyday life. One technique to present abstraction at this level is to provide metaphors, but even this can prove a difficult task. In “A beginner’s guide to forcing” Chow (2009) explains the method developed by Cohen as follows: “Conceptually, this process [of forcing] is analogous to the process of adjoining a new element  $X$  to, say, a given ring  $R$  to obtain a larger ring  $R[X]$ . However, the construction [in forcing] is a lot more complicated . . .” Providing a precise definition for terms such as “ring” would still leave much to be desired for non-mathematicians, as Ring Theory is an entire area of study within Abstract Algebra, and brings with it corresponding notions unfamiliar to many. Writing a case study on a mathematical work for a general audience, then, is a significant challenge, both because of the difficulty entailed in explaining what precisely was accomplished, and in maintaining the reader’s interest. Bearing these difficulties in mind, the rest of this section aims to provide a bit of mathematical background, interwoven with some of the history of CH, for the reader who is unfamiliar with upper-level mathematics in general or Set Theory in particular.

The Continuum Hypothesis concerns the following question: “How many different sets of integers do there exist?” (Gödel, 1947). A detailed history of the hypothesis and its development can be found in Moore (1989). Roughly speaking, in the latter half of the 19th century, Cantor was beginning to make precise the notions of different sizes or *cardinalities* of infinite sets. Two different set sizes, in particular, came up repeatedly. One of these set sizes corresponds to the set of integers, often denoted  $\mathbb{Z} = \{0, 1, -1, 2, -2, 3, -3, \dots\}$ . Given a set such as the integers, one can also talk about a subset of  $\mathbb{Z}$ , meaning: a set whose elements are each integers themselves. For example,  $\{1\}$  is a subset of  $\mathbb{Z}$ ; as are the even integers, denoted  $\{0, 2, -2, 4, -4, 6, -6, \dots\}$ ; the entire set of integers,  $\mathbb{Z}$ ; the empty set (a set with no elements at all); the three element subset  $\{-21, 5, 9\}$ ; and many others. One can even discuss the set of all subsets of  $\mathbb{Z}$ , often called ‘the power set of  $\mathbb{Z}$ ’ and denoted as  $P(\mathbb{Z})$ . In his correspondence with Dedekind, Cantor had showed in a meaningful way that the cardinality of  $\mathbb{Z}$  is strictly smaller than the cardinality of  $P(\mathbb{Z})$ . Naturally, then, the question arose as to whether there is a set whose size lies strictly between that of  $\mathbb{Z}$  and that of  $P(\mathbb{Z})$ .

The idea of *betweenness* depends a great deal on the sort of objects being discussed. Are there any numbers strictly between 0 and 1? If the discussion concerns only integers, then the answer is no. However, if the discussion includes rational numbers (fractions) as well, then the answer is yes. For example,  $\frac{1}{2}$  lies strictly between 0 and 1. Given the inchoate nature of Set Theory at the time, questions about betweenness for sizes of infinite sets were effectively

intractable. Nonetheless, Cantor’s hypothesis was that the answer is no: any infinite subset of  $P(\mathbb{Z})$  has either the same cardinality as  $P(\mathbb{Z})$  or the same cardinality as  $\mathbb{Z}$ ; it could not have some other cardinality lying strictly in between the two. This is the statement referred to throughout as ‘CH.’

How does one prove a result in pure mathematics? From a set-theoretical point of view, one begins with a list of assumptions (also called *axioms*) and rules of logical inference. For many mathematicians, the axiom system used derives from work by Zermelo (1908) and Fraenkel (1922). Their initials give rise to what is known as ZF Set Theory; an additional axiom, known as the Axiom of Choice, is often included as well, adding a C for what is called ZFC Set Theory. The Continuum Hypothesis as described in the preceding paragraph can now be rephrased as follows: In ZF or ZFC Set Theory, can one use the axioms and rules of inference to prove CH is true? Alternatively, can one use the axioms and rules of inference to prove that CH is false? Note that for all the wonderful results that can be proved in these axiom systems, they still have their shortcomings. In particular, though it is widely believed that ZF and ZFC are free of contradictions, one cannot prove this freedom within these formal systems. Additionally, any axiom system<sup>1</sup> strong enough to capture even the most basic notions about arithmetic will have statements that can neither be proved nor disproved. These two results follow from Gödel’s Second and First Incompleteness Theorems, respectively. A mathematical statement that cannot be proved or disproved within an axiom system is said to be *independent* of that axiom system.

Paul Cohen (1963a; 1963b) published “The Independence of the Continuum Hypothesis” in September and November of 1963, approximately a quarter century after Gödel’s earlier paper on CH. Although the Incompleteness Theorems had made it clear that some results would not be provable in ZFC, the Continuum Hypothesis marked the first major statement that fell into such a category. Chronologically speaking, the time from Cantor’s original statement of CH to its inclusion among Hilbert’s problems took about 25 years; from Hilbert’s problems to Gödel’s work took another 35 years; and, finally, from Gödel’s work to Cohen’s development of forcing took another 25 years. Given this eighty-five year span of time, it is noteworthy that Cohen himself wrestled with CH beginning only in 1962, and had developed forcing to resolve the latter half of it by the following year. How was Cohen able to accomplish such a feat so quickly, particularly since he was not a set theorist by trade, and why did no other mathematician (e.g., Gödel) develop an earlier proof?

<sup>1</sup> Strictly speaking, this cannot be *any* axiom system; instead, one must limit to any system of recursively presented axioms (or—as a special case—an axiom system with only finitely many axioms).

## Evolution of Paul Cohen

*. . . in set theory when dealing with fundamental questions, one often has a kind of philosophical basis or conviction, rooted in intuition, which will suggest the technical development of theorems.*

(Cohen, 2002)

*Basically [the Continuum Hypothesis] was not really an enormously involved combinatorial problem; it was a philosophical idea.*

Paul Cohen, July 1985 interview  
(Albers & Alexanderson, 1990)

The Evolving Systems approach to the case study method used throughout this section is primarily derived from the discussion in Gruber & Wallace (1999) and Brower (2003). The former piece includes a total of nine facets that help characterize a case study proceeding with this particular methodology, which originated in work by Howard Gruber as an alternative to the trait-based approaches to Creativity Studies that preceded it (e.g., Guilford, 1950). For the sake of convenience, the facets are listed below in the order in which they will be addressed within this section:

- Facet 1: Uniqueness
- Facet 2: The Epitome
- Facet 7: Problem Solving
- Facet 3: Systems of Belief; Facet 4: Modalities of Thought
- Facet 6: Purposeful Work and Networks of Enterprise

Note that Facet 5: Multiple Timescales, is not specifically addressed, though there is an obvious chronological nature to the discussion here. For further information on the timescales in Cohen's work in particular and on forcing in general, the reader is referred to outside sources (e.g., Moore, 1988; Kanamori, 2008; Cohen, 2002). Even more generally, the reader is referred to these three sources for accounts of the development of forcing from a more mathematical perspective, rather than one with an eye to the Creativity literature. Finally, Facet 8: Contextual Frames and Facet 9: Values, will be covered, to some extent, in subsequent sections.

*Uniqueness*

With regard to Uniqueness, Cohen was like many great mathematicians insofar as he accrued mathematical knowledge in more than one area. Unlike most mathematicians, though, Cohen conducted serious research in many areas. More precisely, he began with early interests in Number Theory and Algebra (e.g., Albers & Alexanderson, 1990), wrote his thesis in Analysis (Cohen, 1958), and was doing work in Differential

Equations (Albers & Alexanderson, 1990) before he took up the set-theoretical problem of CH and other independence related results in 1962 (Cohen, 2002). Conducting research in so many different spheres of mathematics contributed not only to Cohen's knowledge, but also to a variation in the types of problems he was exposed to, which Gruber would refer to as "deviation amplification," and is likely to have led to greater productive novelty (Brower, 2003). In Cohen's own words:

You know, once a problem is solved, I get a little bit bored. I guess that's the price you pay for being a problem solver. I am not really interested in problems that don't seem to stand out . . . I have a mentality of enjoying the challenge of a difficult problem and going directly for it. (Albers & Alexanderson, 1990)

Cohen also comments:

I found it very difficult to settle on one subject. I was interested in number theory—it seemed closest in spirit to problem-solving—but there was actually very little number theory at [the University of Chicago, where I did my graduate work]. The mathematics there was more abstract. I felt that number theory was my first interest, but I wasn't making progress in it. (Albers & Alexanderson, 1990)

This latter quotation segues into Cohen switching to work in Analysis; his results in this area on the Littlewood Conjecture (Cohen, 1960) ultimately won him the top honor in mathematical analysis, the Bôcher Prize, several years later.

Although it will not be explored in depth here, Cohen's ability at this early stage in his career to work in many different areas of mathematics and move from one to another when he felt a lack of progress is akin to the "hierarchical thinking style" described in Sternberg (2010). For the reader interested in this approach to Creative Studies, Cohen is arguably: legislative, local, internal, and liberal. With regard to the legislative, internal, and liberal nature of Cohen's work (i.e., a tendency to do things his own way, work alone, and defy conventions, respectively) much can be gleaned from Cohen's comment: ". . . I am always trying to do things a little bit more primitively. I also wanted to do things completely independently, and that was bad" (Albers & Alexanderson, 1990). On the other hand, whether Cohen's thinking style was more local or global is arguable. It is true that his work on forcing has ultimately been far-reaching, but it developed out of a successful attempt to resolve a single problem. In response to the question of whether or not he views himself as a "problem solver," Cohen remarks:

Yes, I would say that. I'm not particularly proud of it though. I don't think it's a good thing to be, but I don't think I've had much choice . . . I mean,



it's a somewhat, well, egotistical way of being. You know—you want to do one problem. There are other people who have a larger view of mathematics. I would regard it as a higher activity for someone to have a wider perspective from which many new ideas and interactions emerge . . . . But I don't think I had much choice about the kind of mathematician I am. (Albers & Alexanderson, 1990)

It is difficult to assess how accurate this self-characterization is, but at the least one can confidently say that Cohen believed himself to be working on a more local level; that is, problem by problem.

### *Epitome*

The Epitome of this case study (Gruber & Wallace, 1999) is that Paul Cohen was able to develop a new method, forcing, and resolve a long outstanding problem at the forefront of Set Theory. He was able to accomplish this despite his youth and the short time he spent directly engaged with CH, and did so before any of his contemporaries. Gruber and Wallace (1999) suggest that: “In a more extended account the question will be posed, What led up to and what followed from the work in question?” The former part of this question will be at least partly answered, while the latter part (i.e., further mathematical work on forcing) can be found in Kanamori (2008) beginning with the fourth section.

### *Problem Solving*

Given the above discussion of Problem Solving (Gruber & Wallace, 1999) the comparison of Thomas Huxley and Charles Darwin is somewhat appropriate here: the former may have been smarter (in some sense) than the latter, but it was Darwin who was able to adopt a novel perspective and develop his new theory. Similar remarks could be made about Kurt Gödel and Paul Cohen, which will be delved into more deeply later on. Gruber and Wallace (1999) also reference earlier works on Problem Solving, such as the four-stage theory of Wallas (1926) consisting of preparation, incubation, illumination, and verification. Cohen's preparation will be returned to later as well. The incubation period for Cohen came during a road-trip he took with his new wife near the Grand Canyon, at which point he had the sort of illuminative insight that is sometimes associated with Mathematical conclusions (e.g., Poincaré, H., Russell, B., & Maitland, F., 1914). Of this time period, Cohen remarks:

There are certainly moments in any mathematical discovery when the resolution of a problem takes place at such a subconscious level that, in retrospect, it seems impossible to dissect it and explains its origin. Rather, the entire idea presents itself at once,

often perhaps in a vague form, but gradually become more precise. (Cohen, 2002)

Investigating the conditions directly preceding this sort of insight are inherent to Gruber's case study methodology; such a focus on “microgenesis” is addressed in, for example, Brower (2003). Cohen himself details this precise time in his speech at the 2006 Gödel Centennial in Vienna<sup>2</sup> as transcribed below:

Rather early in the game, I think forcing occurred in a very, very easy form to me, but I didn't know what I had. And I told this story somewhere: I met my wife at that time and we took a long trip. I wanted to show her the United States—she had come from Sweden—we took a long trip to the Grand Canyon, to various national parks, and as I was driving I did a very dangerous thing: I thought hard about mathematics the whole time. And during that time the idea became clearer.

Though Cohen was quickly convinced his own ideas were correct, the verification process took him all the way to the East Coast. Unsatisfied with the response from fellow mathematicians at Stanford and Berkeley, Cohen left for the Institute for Advanced Study (Moore, 1988). His reasoning can be inferred from a letter sent to Gödel at this time, on May 6, 1963, in which Cohen writes:

In short, what I am trying to say is that only you, with your pre-eminent position in the field, can give the ‘stamp of approval’ which I would so much desire. I hope very much that you can study the manuscript thoroughly and by next weekend be willing to discuss it in more detail.

At this point, it is perhaps worthwhile to connect from the Creativity literature on Problem Solving in general to the Mathematics Education literature on Mathematical Problem Solving in particular. The seminal piece with regard to the latter is George Polya's “How to solve it” (1945) in which he provides an encyclopedic listing of heuristics (i.e., strategies for solving mathematical problems); incidentally, Polya and Cohen were both colleagues and good friends at Stanford. Later on, Schoenfeld (1985) developed a framework for what is necessary in order to be highly skilled at Mathematical Problem Solving: resources, heuristics, control, and belief systems. A more detailed investigation might explore how the framework of Schoenfeld relates to the methodology of Gruber, with many of the four areas above subsuming facets of the latter. Roughly speaking, resources refers to the mathematical knowledge accumulated by Cohen; heuristics refers to the strategies he used in tackling CH; control refers to Cohen's metacognitive ability to self-regulate within

<sup>2</sup> A recording is available online at <http://www.youtube.com/watch?feature=endscreen&v=l-8KzD2U9J4&NR=1>.



the problem solving process, e.g., to abandon seemingly unfruitful approaches and persist with those that had potential; and belief systems refers in part to Cohen's mathematical worldview as he developed the method of forcing. Resources and control will be touched upon in the later discussion of Cohen's mathematical preparation; heuristics and beliefs will be implicit in the next paragraph on Cohen's Systems of Belief and Modalities of Thought.

### *Systems of Belief and Modality of Thought*

Cohen's Systems of Belief (Gruber & Wallace, 1999) evolved over time during the development of the technique of forcing. Initially, Cohen was primarily interested in resolving questions about the integers, which he believed could be done through a sort of "decision procedure" approach. Subsequently, he took on a newfound interest in philosophical arguments, and, when finally exposed to CH, Cohen believed an attack would be best mounted using set-theoretical "models." The term "decision procedures" refers to an algorithmic process to resolve a mathematical question. For example, when asked about the addition of integers, there is a procedural way of answering: a calculator can even be programmed to resolve such questions. (Issues about memory capacity are unimportant here; the salient point is that there is a way, at least in theory, of adding up any finite number of integers.) Since number theoretical questions can be expressed purely symbolically in the language of Set Theory, one might wonder: Could there exist, in theory, a calculator-like device that takes as input the symbols and gives as output a proof (if the statement is true) or refutation (if the statement is false)? As referenced earlier in this paper, there are certain statements—among them CH—that cannot be proved or refuted within any axiom system<sup>3</sup> that captures even the notions from basic arithmetic. Thus, no such proof-calculator can exist. However, Cohen was initially unaware of this result, proved first by Gödel and later included in a textbook by Stephen Kleene. In his own words, Cohen remarks:

Because of my interest in number theory, however, I did become spontaneously interested in the idea of finding a decision procedure for certain identities . . . . I saw that the first problem would be to develop some kind of formal system and then make an inductive analysis of the complexity of statements. In a remarkable twist this crude idea was to resurface in the method of 'forcing' that I invented in my proof of the independence of the continuum hypothesis. (Albers & Alexanderson, 1990)

Cohen's rough idea was: to figure out how *complex* different mathematical statements are, begin by describing how to decide the simplest ones, and then build up slowly so as to be able to decide more complicated ones. That such a belief seems reasonable makes Gödel's Incompleteness result all the more surprising. Quoting Cohen again:

At the time these ideas were not clearly formulated in my mind, but they grew and grew and I thought, well, let's see—if you actually wrote down the rules of deduction—why couldn't you in principle get a decision procedure? I had in mind a kind of procedure which would gradually reduce statements to simpler and simpler statements. I met a few logicians at Chicago and told them about my ideas. One of them, a graduate student too, said, "You certainly can't get a decision procedure for even such a limited class of problems, because that would contradict Gödel's theorem." He wasn't too sure of the details, so he wasn't able to convince me by his arguments, but he said, "Why don't you read Kleene's book, *Metamathematics*?" (Albers & Alexanderson, 1990)

Shortly thereafter, Kleene came to the University of Chicago, where the graduate student Paul Cohen asked him directly whether or not these ideas about a decision procedure contradicted Gödel's work. Kleene assured him they did, and so Cohen began to read Gödel's work (Cohen, 2008). Cohen was not particularly fond of the writing style he encountered: "Gödel often expressed his ideas in rather convoluted ways and was concerned with philosophical nuances, which I in all honesty have never found interesting" (Cohen, 2002). Nevertheless, he began to realize that despite his own personal distaste for so-called philosophical approaches, such arguments could indeed have ramifications within Number Theory. This conclusion was to the surprise of Cohen, as discussed in the 2006 Gödel Centennial Address and summarized in Cohen (2008) as: "How can someone [i.e., Gödel] thinking about logic in almost philosophical terms discover a result that had implications for Diophantine [i.e., number theoretical] equations?" As Cohen took on the belief that such philosophical approaches might be worthwhile, he combined this with another belief: set-theoretical models were to be essential in tackling CH.

Roughly speaking, a *model* is an interpretation of a collection of axioms in which they all hold true. For an introduction to models and set theory, see Wilder (1965). The axioms of Euclidean geometry, for example, can be interpreted with the sort of model one learns in secondary school, where points correspond to *dots* in the plane, and so forth. In a similar vein, the axioms of ZF or ZFC Set Theory can be thought of in terms of a model; in fact, there are many different models and interpretations for this collection

<sup>3</sup> An earlier remark applies here as well. Strictly speaking, one should not consider *any* axiom system, but rather any recursively presented axiom system.

of axioms. Of his move to thinking in terms of models, and especially a relevant result known as the Skolem-Lowenheim Theorem, Cohen states: “I felt elated yet also very discouraged . . . the feeling of elation was that I had eliminated many wrong possibilities by totally deserting the proof-theoretic approach. I was back in mathematics, not in philosophy” (Cohen, 2002). Nevertheless, the idea that there are multiple interpretations for a single set of axioms, particularly the set taken as underlying all of mathematics, is perhaps confusing to the non-mathematician. In fact, this idea is just as confusing (if not more so) to mathematicians, and was even difficult for Cohen to face:

. . . the existence of *many* possible models of mathematics is difficult to accept upon first encounter, so that a possible reaction may very well be that somehow axiomatic set theory does not correspond to an intuitive picture of the mathematical universe, and that these results are not really part of normal mathematics . . . I can assure you that, even in my own work, one of the most difficult parts of proving independence results was to overcome the psychological fear of thinking about the existence of various models of set theory as being natural objects in mathematics about which one could use natural mathematical intuition. (Cohen, 2002)

With regard to the ways in which creative people think, one might wonder how mathematicians in particular think when developing their creative works. Perhaps one may wonder: “Do mathematicians think in equations?” (Gruber & Wallace, 1999). More specifically, one could ask: Do geometers think visually? Do logicians think symbolically? This more general idea of how one thinks (visually, verbally, symbolically, etc.) is referred to as one’s Modality of Thought (Gruber & Wallace, 1999). For Cohen in particular, his way of thinking was with the aforementioned models. Of approaching Set Theory in general, he writes, “we will never speak about proofs but only about models” (Cohen, 2002). With this Modality of Thought, Cohen was able to adopt the belief that model construction was a key feature in devising his proof. Still, his earlier philosophical beliefs were retained to some extent as well:

Of course, in the final form, it is very difficult to separate what is theoretic and what is syntactical. As I struggled to make these ideas precise, I vacillated between two approaches: the model theoretic, which I regarded as roughly more mathematical, and the syntactical-forcing, which I thought as more philosophical. (Cohen, 2008)

Fundamentally, the development of forcing was made possible by the adoption of these two beliefs: the idea of coming up with an interpretation, or model, of the Set Theory axioms in which CH does not hold; and the idea of building up this

model by adding new pieces to the model and examining them, in increasing order of complexity, to ensure that all the axioms continued to be true, but that the final model included a new set size between that of  $\mathbb{Z}$  and that of  $P(\mathbb{Z})$ .

#### *Purposeful Work and Network of Enterprises*

Cohen’s purpose in his mathematical work was to seek simple answers for difficult problems. Of the Skolem-Lowenheim Theorem, mentioned earlier as relating to models and being crucial to the resolution of CH, Cohen remarked it to be “perhaps a typical example of how a fundamental result which has such wide application must of necessity be simple” (Cohen, 2002). Of his own work, Cohen declared his “ideal” to be “to take a problem which looks very complicated and find a simple solution” (Albers & Alexanderson, 1990). An essential feature of the Evolving Systems approach is that the purposeful work be carried out over a long period of time—perhaps even one’s entire life (Gruber & Wallace, 1999). Cohen was already aware of Galois Theory, a subject for an upper-level college or first-year graduate course in Mathematics, by the age of around nine (Albers & Alexanderson, 1990). His “earliest interest in logic” dates back to his time at Stuyvesant High School (Cohen, 2008). Though not working explicitly on CH or even in Set Theory, Cohen spent about two decades engaged in serious mathematical thought before having his breakthrough insight during a trip to the Grand Canyon. Descriptions of Cohen’s various mathematical achievements and work over time can be found in several of the references provided; this paper aims only to discuss one specific aspect of his development, as it relates to a particular work post-dating the earlier accounts of Cohen and forcing.

During his commentary at the 2006 Gödel Centennial, Cohen briefly touched upon a particular problem that was resolved five years before his work on CH commenced. Known as the Post Problem and solved by Richard Friedberg, Cohen states:

At that time there was great interest among Raymond [Smullyan] and some other people about the Post Problem. And that’s a problem which could have interested me; it had a mathematical flavor to it. But I never thought about it, and occasionally we’d have coffee and I’d hear these people talk about it. But one day, someone came to my office and said, “This problem’s been solved.” And I said, “Really?” “Yes, here’s the letter. I can’t believe it’s true!” And he gave it to me and I read it. I went to the blackboard, took some chalk, and I said, “Well, it seems right.” This is the proof by Friedberg—and so that was my only contact with logic at that point. But I still never lost this idea of somehow thinking about the foundations of mathematics: trying to find some kind

of inductive technique for simplifying propositions; perhaps leading to a decision procedure, when impossible.<sup>4</sup>

An interesting feature of the Post Problem is that the technique used to resolve it, known as the priority method, is one of only a couple “important precursors to the modern theory of forcing” (Kunen, 1980). This observation was made early on by mathematical logician Georg Kreisel, who “saw an analogy between forcing and Friedberg’s [1957] priority argument” and noted as much in a letter to Gödel in April of 1963 (Moore, 1988). Cohen’s remarks above were later written up as a preface to a re-printing of “Set Theory and the Continuum Hypothesis” (Cohen, 2008) in which the section corresponding to the above transcription is:

A small group of students were very interested in Emil Post’s problem about maximal degree of unsolvability. I did dally with the thought of working on it, but in the end did not. Suddenly, one day a letter arrived containing a sketch of the solution by Richard Friedberg (Friedberg, 1957), and it was brought to my office. Amidst a certain degree of skepticism, I checked the proof and could find nothing wrong. It was exactly the kind of thing I would like to have done. I mentally resolved that I would not let an opportunity like that pass me again.

It is not easy to know how accurately Cohen, in this 2006 recollection or his earlier 2002 piece, is recounting the events from four decades earlier. Even if true, one’s own stories can themselves be a form of creative work: depending on what is included or omitted, and what is viewed as being continuous (or not) with earlier and subsequent events (Bateson, 2001). Of course, *post hoc* recollections of inspiration are often less accurate (Gruber, 1981). In any event, taking Cohen’s recollections at face value, a few observations are of note. First, Cohen is deeply interested in a problem whose method of solution is closely related to his subsequent work on forcing.<sup>5</sup> Second, Cohen has received only a sketch of a proof: as is nearly always the case, mathematical works at a high-level (especially pre-prints) will leave at least some of the details to the reader. Cohen’s initial response, then, is to walk up to a blackboard—chalk in hand—and “check the proof” to see if it is correct. That Cohen would be able to work through these details and understand such a method is especially notable insofar as it takes place a half-decade (1957) before he begins to wrestle with CH (1962). Third, Cohen promises himself that he will not let a problem like that “pass him again.” This hones his sense of purpose, as Cohen

is now determined not to let similar problems pass him by in the future; indeed, he stays true to this resolution. Thus, while there are stories of Cohen asking logicians in 1962 for the toughest problem, being told it is CH, and resolving it by the next year (e.g., his introduction at the Gödel Centennial) it is clear that Cohen’s background—in terms of the beliefs and mode of thought described earlier, as well as his familiarity with the Post Problem and its solution—prepared him at least five years before his work on forcing commenced.

As a segue into the next section, one must also take note of the various persons with whom Cohen came into contact. From reading the earlier work of Skolem, to receiving a preprint from Friedberg, to interacting with several logicians—including Gödel, once the key ideas of forcing were in place—Cohen was able to work in many areas of mathematics and communicate with many mathematicians. This Network of Enterprises (Gruber & Wallace, 1999) further typifies many creative individuals, allowing for deviation amplification in addition to the acquisition of diverse sorts of knowledge. In exploring such a concept, even within an ivory tower discipline such as pure mathematics, one must step outside of the individual and investigate what is occurring from a societal and cultural perspective (Csikszentmihalyi, 1988).

#### Sociocultural Factors

There are a total of five Contextual Frames (Gruber & Wallace, 1999) or contexts that are singled out as relevant to the individual being studied with the Evolving Systems approach. In the case of Cohen, these are: enterprises directly related to forcing and its development; his body of work and overall purpose; his family; his “professional milieu,” i.e., fellow mathematicians; and the “sociohistorical milieu.” Of these, only the final two will be delved into below. The first two have already been discussed to some extent; meanwhile, with regard to Cohen’s family, it was mentioned earlier that he was driving with his wife, shortly after they had married, when his thoughts on forcing finally coalesced. The two would also go on to have several children, among whom one, Charles, provides a poignant contribution to the American Mathematical Society notes on his father’s passing (Cohen, 2010). Nevertheless, it is the mathematicians who were around Paul Cohen and the sociohistorical milieu of the time that warrant further attention here.

How are sociocultural factors related to creative developments in pure mathematics? Cohen, in his recollections about his early childhood, remarks:

[M]ath especially appealed to me. If you read something about electricity, for instance, you find out that you need a lab to do anything yourself, but with math you can do problems right away.

<sup>4</sup> Video of Cohen’s speech can be viewed at <http://www.youtube.com/watch?v=1qSSZqzfY9U&t=7m46s>

<sup>5</sup> For further discussion of the similarities, see <http://mathoverflow.net/questions/124011/similarities-between-posts-problem-and-cohens-forcing>.



So I just naturally went further in math. (Albers & Alexanderson, 1990)

With no need for a laboratory or any equipment, one might think that mathematical developments—even great ones—could come about without societal or cultural concerns. Of course, social network is important, even in mathematics, and Cohen was in contact with several strong mathematicians. In addition to those already mentioned Cohen was also in contact with several logicians, including Azriel Levy and Solomon Feferman. This is necessary for discussing mathematical ideas, and also for obtaining mathematical works, particularly in the pre-Internet era, such as Friedberg's sketch of an argument that resembled Cohen's subsequent technique of forcing. Many results in logic in particular were known among a small network of people, but not written down anywhere accessible. Historian of mathematics Moore (1988) notes: "At the time [around which forcing was developed] there existed a substantial 'folklore'—a body of unpublished results that were known only to the so-called *cognoscenti* . . ." The social connections afforded Cohen, then, were an integral part of his subsequent work.

Additionally, there is the consideration of which mathematical questions are deemed important enough to warrant the attention of mathematicians. For the case at hand, it was Hilbert's list of 23 problems at the turn of the century that lent credence to the notion that CH was a problem worthy of one's time. In particular, CH joined a long list of famous mathematical problems that were determined by the *field* of mathematicians to be important within the *domain* of mathematics (Csikszentmihalyi, 1999). Even still, Cohen comments:

No one specifically said so, but there was a feeling that something radically new would have to be done to solve [the Continuum Hypothesis]. I didn't get the impression that mathematicians not in logic were all that interested. That may sound strange, but it seemed true at the time. All in all, the problem seemed to be in a kind of limbo. (Albers & Alexanderson, 1990)

This general "feeling" pairs with Hilbert's list to produce a problem that was at once considered important enough to attract Cohen, yet difficult enough to dissuade other mathematicians from working on it.

Separately, there is a sociohistorical question regarding what precisely Cohen proved. The notion of proving the *independence* of CH requires proof that it is independent from a particular set of axioms. As remarked in the second section, Cohen's work demonstrated that ZF and ZFC Set Theory (assuming they are free of contradictions) are not powerful enough to prove or disprove CH. Though the problem was initially posed in 1878, it was not until 1922 that ZF Set Theory had been formally proposed, and even then it would

not be immediate for mathematicians to latch on to it. Thus, for the first 45 years of the problem, the work that was done by Cohen was, historically speaking, not possible. From a societal viewpoint, mathematicians had to accept ZF or ZFC Set Theory as somehow capturing mathematics accurately for Cohen's result to be of any interest. Sociocultural and sociohistorical factors would continue to be relevant even after Cohen's result was published and accepted. In particular, Cohen had demonstrated something about the *provability* of a statement; he had not settled the separate question of whether CH is actually true or false. For those mathematicians who conceive of a *true mathematics* existing, regardless of the set-theoretical axiom system ultimately used, there is a lingering dissatisfaction insofar as CH was only shown not to follow from ZFC Set Theory. Cohen addressed this in his 2002 and 2008 works where he argues that, ultimately, CH is false; Gödel (1947) suggested a search for another intuitive axiom to be added on to ZFC Set Theory: one that accords with mathematicians' natural intuition, and which can settle CH one way or the other. Regardless, these concerns continue today, and are in large part determined not strictly by some mathematical fact, but rather by the views of mathematicians—set theorists, logicians, and otherwise—who work in mathematics.

#### *Ramifications in Mathematics Education and Creativity Studies*

The final facet of the Evolving Systems methodology to be considered is that of Values (Gruber & Wallace, 1999). In particular, Gruber and Wallace state: "For better or worse . . . certain aspects of creative work have been neglected—by us and by like-minded colleagues." Although they go on to discuss more specific issues surrounding morality, one might wonder whether certain *areas* of creative work are being neglected when it comes to conducting case studies. What are the ramifications given a literature on creativity in which, for example, pure mathematics receives little attention? The difficulties of carrying out such a case study given a non-mathematical readership were discussed early on; even still, this paper is sure to have covered some topics without complete clarity (e.g., what exactly is a *model*?) and to have avoided deeper discussions of others (the astute reader has no doubt noticed that forcing is nowhere defined here, nor are the specific axioms of ZFC listed). To shy away from pursuing case studies in mathematical creativity is dangerous for several reasons. Here are but two: first, it may leave others with the impression that creativity is absent from mathematics; second, it may deprive those who wish to learn and teach mathematics from the insights that can be gleaned from better understanding how a great mathematical work comes about. This latter point is expanded upon below.

What are the implications of a single case study on a particular mathematical development in a broader context? In an ideal world, to learn from Cohen's approach some new way of fostering mathematical creativity. This paper has argued that Cohen was specifically primed to resolve CH by virtue of his initial interests in decision procedures, and his subsequent Modality of Thought being rooted in the use of set-theoretical models to think about mathematics. From these two observations alone, certain questions arise. The approach using decision procedures was mathematically ungrounded, as it violated Gödel's Incompleteness result, which eventually led those in Cohen's social network to inform him of his error. This then led Cohen to read Gödel's work (first in Kleene's textbook) and his realization that philosophical arguments could be applied successfully to number theoretical questions. However, it was his *incorrect* initial approach that re-appeared in his ultimate solution. Perhaps other logicians around the time of Gödel (including Gödel himself) were less inclined to think in this particular way, as they were already aware that it was, in a strong sense, a dead end. How does this inform the way that teachers of mathematics respond to students who are attempting to solve problems in ways that are easily seen as "incorrect?"<sup>6</sup> Pedagogically speaking, how should educators deal with these approaches? Should they correct the students? If so, when? Surely the first observation leads to more questions than answers.

Similarly, what does the idea of a Modality of Thought suggest to teachers of mathematics? Gruber and Wallace (1999) wondered whether mathematicians thought in terms of equations. For mathematics students who think visually: should they be encouraged to pursue areas of Geometry or Topology? For those who think symbolically: should they be encouraged to pursue Algebra or Logic? What general lessons, if any, can be drawn from Cohen's success in using his model-based thinking to dispose of a great, unsolved problem in Set Theory? Again, more questions than answers arise, though it would be rather ironic to use a case study on Paul Cohen to justify pointing students in certain directions: after all, he was renowned for the vast number of areas within mathematics in which he worked. Perhaps the take-home message of this early foray into case studies on highly creative mathematical works is that there are further conversations to be had, even if the specific result serving as a springboard—here, the technique of forcing—remains as a bit of a black-box.

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<sup>6</sup> An interesting piece on partial progress by Fields Medalist and MacArthur Fellow Terence Tao is relevant to the discussion here. As Tao remarks: "it can often be profitable to try a technique on a problem even if you know in advance that it cannot possibly solve the problem completely." The reader is referred to <https://plus.google.com/114134834346472219368/posts/Xdm8eiPLWZp> for the full text.

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