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Fostering Mathematical Creativity through Problem Posing and Modeling using Dynamic Geometry: Viviani's Problem in the Classroom

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This paper discusses a classroom experience in which a group of prospective secondary mathematics teachers were asked to create, cooperatively (in class) and individually, problems related to Viviani's problem using a problem-posing framework. When appropriate, students used Sketchpad to explore the problem to better understand its attributes (e.g., knowns, unknowns, and restrictions) and model its solution. With support and guidance, each student was able to create at least one very interesting and good mathematical problem.

Keywords: mathematical creativity, problem posing, problem-posing framework, Viviani's problem, modeling, dynamic geometry

Introduction

Mathematical creativity is essential to the continued development of mathematical knowledge, which grows by creating new mathematical constructs (e.g., the concept of infinitesimal), definitions (e.g., the definition of infinite sets), axioms (e.g., the axiom of choice), algorithms (e.g., the Euclidean algorithm to find the GCD of two integers), formulas (e.g., Euler's polyhedral formula), theorems (e.g., Fermat's last theorem) and problems (e.g., the four-color problem).

One of the goals of mathematics teaching at all levels should be to foster students' mathematical creativity (Polya, 1945; Sriraman, 2004). This goal stands in stark contrast with some current instructional practices that encourage students to memorize formulas so they can apply them to solve a narrow range of prescribed routine problems. However, one of the first obstacles that teachers face as we try to nurture students' mathematical creativity is to define and operationalize this complex construct. To begin with, there is not a universally accepted definition of mathematical creativity. To implement the goal of fostering the mathematical creativity of my students, prospective secondary mathematics teachers, I decided to follow Hadamard's (1945) claim that "between the work of a student who tries to solve a difficult problem in geometry or algebra and a work of invention (creation) there is only a difference of degree" (p. 104). Encouraged by the literature on problem posing and creativity, I provided my students with opportunities to create their own problems to pursue, rather than giving them pre-formulated problems.

Problem Posing as an Attribute of Creativity

Posing problems has long been used as a tool to investigate creativity. Silver (1994) refers to problem posing as "both the generation of new problems and the reformulation of given problems" (p. 19). Thus, we may engage in problem posing before, during, and after solving a problem. Some researchers and practitioners have highlighted the ability to pose problems as a characteristic of creativity not only in mathematics but also in other domains. For example, Getzels and Csikszentmihalyi (1976) identified problem finding as a key indicator of artistic creativity. The prominent French mathematician Hadamard (1945) claimed that an attribute of mathematical creativity was the aptitude to pose crucial mathematical research questions or problems. Regarding mathematically gifted students, Greenes (1981) found that a critical feature of such students was their capacity to spontaneously pose problems. In his study of the mathematical ability of gifted students, Krutetskii (1976) found that an indicator of mathematical talent was the ability to formulate problems from given information. In a more recent study, Ellerton (1986) used problem posing as a tool to study mathematically talented school children. She found that the more capable students posed problems of greater complexity than those less capable. Getzels and Jackson (1962) have also included problem-posing tasks in instruments aimed at measuring mathematical creativity. These researchers found that the more able students posed problems requiring more complex *solutions* than the problems posed by others.

In this paper I describe a classroom experience in problem posing with a group of prospective secondary mathematics teachers whose goal was to examine whether a problem-posing framework and Dynamic Geometry could help students generate newly related problems to a given problem. The students were enrolled in a college geometry course that I taught. Since I focus on the problem-posing process, the proofs will not be discussed.

Background

Previously, the class had investigated the Varignon theorem that I reformulated as a problem to provide students with a diverse range of opportunities to pose and solve problems. The version of the problem that the class started with was as follows: Let $ABCD$ be a parallelogram and E, F, G, and H be the midpoints of its consecutive sides. What type of quadrilateral is $EFGH$? ($EFGH$ is known as the medial quadrilateral in Figure 1.)

The class then modified the attributes of this version of the Varignon problem and investigated the following types of problems: proof, special, general, extended, and converse problems, as suggested by the problem-posing framework depicted in Figure 2. For further details of the investigation, see Contreras (2009, in press). The investigation was facilitated by The Geometer's Sketchpad (GSP, Jackiw, 2001), but other types of interactive geometry software such as Cabri Geometry (Texas Instruments, 1998) or Geogebra (Hohenwarter, 2002) could achieve the same goal.

To start a new investigation, I challenged the class with an open-ended generalization of a fruitful problem known as Viviani's problem.

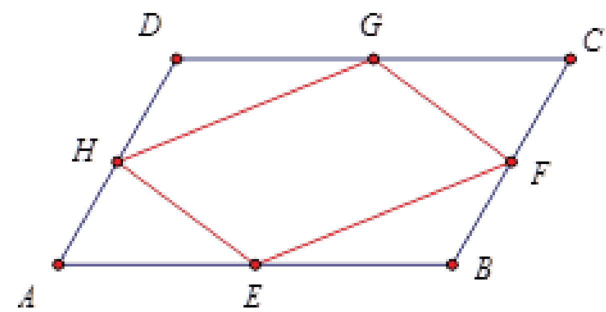

Figure 1. The medial quadrilateral of a parallelogram

An Open-Ended Generalization of Viviani's Problem

The problem with which the class started the investigation was formulated as follows:

> A campsite lies along the base of an isosceles triangular region. A camper wants to set up her tent so that the sum of the distances to the other two sides is minimal (See Figure 3). Where should the tent be placed? Interpret geometrically the sum of such distances.

The students, accustomed to problem-based learning, wanted to immediately rush to construct the configuration with GSP. However, before they did so, I asked them to guess the location of the optimal point. Most members of the class claimed that the midpoint of the base was the required point.

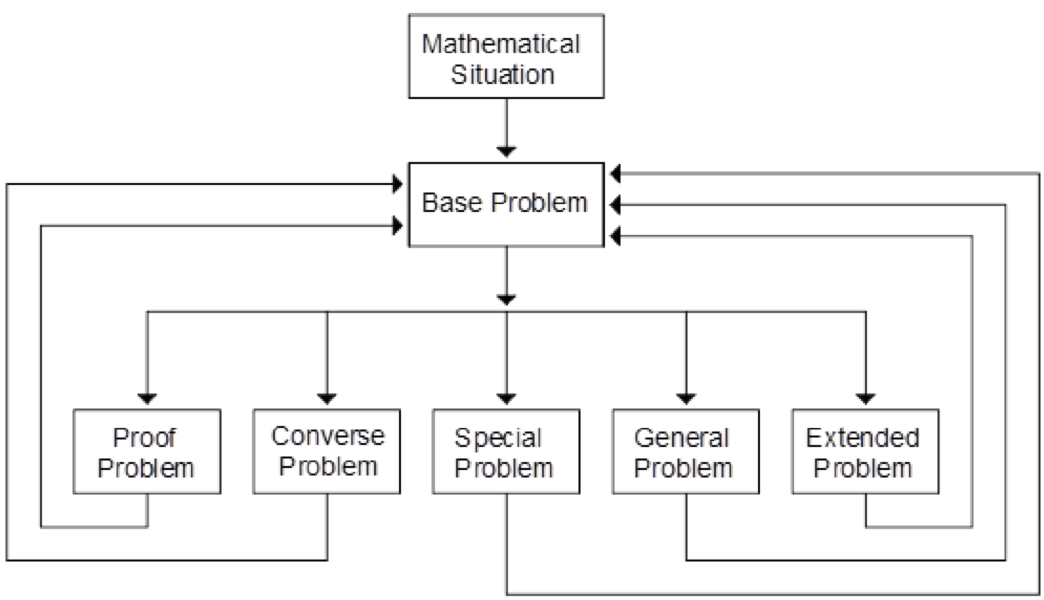

Figure 2. A problem-posing framework

After students had constructed the configuration, they were surprised that any point on the base of the isosceles triangle satisfied the required condition. Interpreting geometrically the distance $DE + DF$ was a little bit more challenging, but after dragging point *D* to an extreme location (e.g., vertex *A* or *B*), most of them noticed that $DE + DF$ is the length of any of the altitudes of ΔABC corresponding to the congruent sides (Figure 4). Thus, another unrelated problem to the original was found (i.e., prove that the altitudes corresponding to the congruent sides of an isosceles triangle are congruent).

After the members of the class had generated a conjecture, a student spontaneously challenged the class with the question "How do we prove this?" Because I wanted to emphasize the problem-posing process, I asked the student to reformulate his problem in self-contained form (i.e., a complete formal problem statement that includes all the known and unknown attributes). After some discussion, the class accepted the reformulation of the problem as displayed in Table 1, Problem 1. After providing students with a strategic hint, they were able to develop a proof on their own involving the diagram shown in Figure 4.

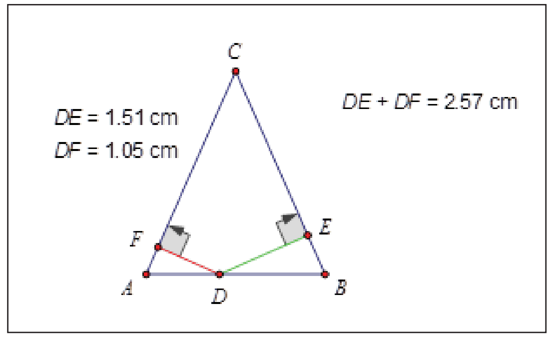

Figure 3. Locate D such that $DE + DF$ is minimal

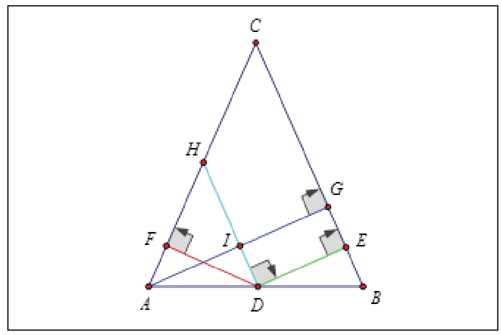

Figure 4. Diagram used to solve the initial problem

Students as Creators of Problems

I then asked students to create additional problems. One student proposed a converse problem while another formulated a special problem and a general problem. The initial converse problem was as follows: If $DE + DF = AG$, then ΔABC is isosceles (Figure 4). Notice that this statement was not expressed in the form of a problem, but rather in form of a conjecture or theorem since it was stated as a definitive assertion. After students explored the assertion to get acquainted with it, they proved it. Then I asked the class to reformulate the initial converse problem, the final version of this problem is displayed in Table 2, Problem 2.

Initial (combined) special and general problems created by another student were expressed in the following terms: What if the triangle is equilateral or scalene? The class first investigated the case where ΔABC is equilateral (Figure 5). One student argued that the location of the tent was still on side $\overline{AB}$ because that was the campsite, while another student contended that the camping ground is on the perimeter of the triangle because the base of an equilateral triangle is any of the three sides since they are congruent. After some discussion the class agreed on the last interpretation and proceeded to find the solution to the problem. Here is the reasoning of one of the students:

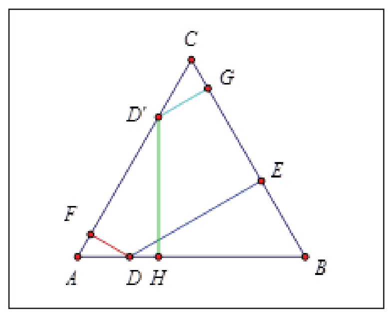

Figure 5. The location of the tent is on any side when ΔABC is equilateral

> If the tent (point *D*) is located on $\overline{AB}$, then the sum of the distances (to the other two sides) is equal to the altitude of *A* or *B* because those two altitudes are equal. If we set up the tent on side $\overline{AC}$, then the sum is also equal to the altitude of *C* because the altitudes of *A* and *C* are equal. So you can put it on any side of the triangle.

To enhance the students' problem-posing experience, I challenged the class with the questions "What if the campsite is now the whole equilateral triangle? Where should we set up the tent so that the sum of the distances to the three sides of the triangle is minimal? Most students conjectured that the tent should be set up on the perimeter of the equilateral triangle. They were delighted to discover, using GSP, that any point of the interior of the triangle also minimizes the sum and that this sum is equal to the length of the altitude of the equilateral triangle. The class developed an analytic proof on their own, and I suggested an invariant (area) proof that the

Table 1. Problems Created in Class

1. Let D be any point on the base of an isosceles triangle ABC with $\overline{AC} \cong \overline{BC}$. Prove that the sum of distances from D to sides AC and BC is the length of the altitude that goes through the vertex of one of the congruent angles.
2. Let D be an arbitrary point on side $\overline{AB}$ of a triangle ABC. Construct the feet of the perpendiculars from D to the other two sides of ΔABC, E and F respectively. Let G be the foot of the altitude of ΔABC corresponding to vertex A. If $AG = DE + DF$, prove that ΔABC is isosceles.
3. Construct the distances from any point located in the interior of an equilateral triangle, or on the triangle, to its sides. Prove that the sum of these three distances is minimal and equal to the length of its altitude.
4. Find all points for which the sum of their distances to the sides of a scalene triangle is minimal.

Table 2. Problems Created Individually

1. Four towns are connected with a system of four roads that form a square. The mayors of the cities want to build a recreation park and connect it to each road. Describe all the possible locations of the recreational park such that the sum of the four distances is minimal.
2. Find all the points such that their sum of their distances to the sides of a parallelogram is as small as possible. Interpret geometrically the sum of the four distances.
3. If the sum of the distances from any [every] interior point of a triangle is a constant, is the triangle necessarily equilateral? Justify your response.
4. Let P be a point in the interior of a regular hexagon. Prove that the sum of its distances to the sides of the hexagon is a constant. If possible, interpret the constant geometrically.
5. P is an arbitrary point in the interior of a triangle ABC and PD, PE and PF are the distances from the point to the sides of the triangle. Find the location of the point for which $PD \times PE \times PF$ is a maximum.
6. Four cities located at the vertices of a quadrilateral want to construct a power-generating plant. To keep the cost of the cables and towers at a minimum, the plant needs to be built at a location so that the sum of the distances from the plant to the four cities is as small as possible. Find the optimal location of the plant.
7. Three major cities want to construct a rest area at a place that has the same distance from the three highways that connect the cities. Describe how to find all the possible places.
8. A parking area has the shape of an isosceles triangle. Where should you park if you want to minimize the sum of the distances to the three sides of the triangle?

class completed with some tactical hints. The final version of the problem is displayed in Table 1, Problem 3. This is the problem known as Viviani's problem.

The students then returned to the investigation of the general problem. After some further discussion the class agreed to reformulate the initial general problem (what if the triangle is scalene?) as shown in Table 1, Problem 4. The students used GSP to provide a conjecture, which they then proved with appropriate hints.

Continuing the Problem-Posing Experience

As a midterm project, students were asked to produce a new problem related to any of their created problems and to use GSP, if appropriate. Afterward they were expected to make a conjecture about its solution and then, if possible, justify the conjecture mathematically. Accustomed to posing problems, students knew that a new problem meant a problem that we had not discussed in class, read in a book, or learned from another source. Because I did not require students to completely solve the problem, which in most cases involved developing a proof, students were willing to take risks in their problem-posing endeavors that they otherwise would have not. One concern in problem-posing tasks is that students may formulate only problems that they have confidence in solving (Silver, Mamona-Downs, Leung, & Kenney, 1996). If so, then students' problem posing may reflect an ability to solve problems rather than a capability to create them.

I provided students with some support in the form of potential avenues to find problems without being too directive or helpful. Also, I advised students to use GSP or other appropriate tools to explore the potential new problem to make sure that it was a well-posed problem. Students knew that the modification of an attribute sometimes required changing another attribute to avoid creating problems that obviously made no sense.

As a first step in their independent problem-posing adventures, students underlined the attributes of a problem that could be changed. We called the initial problem a base problem. Thus, Problems 2–4 displayed in Table 1 are problems related to (the base) Problem 1. In turn, we can now take any of these new problems and create offshoots of these problems. Some members of the class used Problem 3 as their base problem. They underlined the following attributes as potential changes to create their new problems and provided the possible alternatives indicated in parenthesis: interior (exterior), equilateral (equiangular, scalene, and regular), triangle (other quadrilaterals and polygons), sum (product, quotient, difference), sides (vertices), and minimal (maximal, constant). Other students remained using Problem 1 as their base problem. All of the students knew that they could reverse known and unknown attributes to pose a converse problem (i.e., reverse problem). Having more experience in problem posing, I also suggested considering analogous three-dimensional cases. I asked students to use a variety of real world contexts and problem format (open-ended or closed-ended).

Before presenting their problems to the class, students peer-reviewed each other's problems in small groups. They were asked to focus on mathematical content, clarity, and language. While some students presented their created problems and solutions to the class, others only presented them with tentative conjectures about their solutions. During the presentation, the rest of the class offered additional suggestions to improve the formulation of some of the problems. Table 2 depicts the final version of the problems that students created for the midterm project. I subjectively judged each problem as being well-posed, non-trivial, and good with a solution that is mathematically interesting, beautiful, and surprising.

Students' Learning in Inquiry, Dynamic Geometry Environments

In my experience, inquiry-oriented mathematics learning which includes posing and solving problems within dynamic geometry environments can help students develop some mathematical habits of mind that are recommended by several organizations or associations such as the National Council of Teachers of Mathematics (1989, 2000), the National Research Council (2001), and the Common Core State Standards Initiative (CCSSI, 2010). Specifically, as the students were involved in the process of posing and, sometimes, solving problems related to Viviani's problem, they were engaged in a plethora of Mathematical Practices advocated by CCSSI:

- *Making sense of problems*. When modifying the attributes of a problem, students analyzed the structure of the problem (e.g., givens, constraints, relationships, and goals) to create related problems that were non-trivial and well-posed rather than just jumping into an unreflective attempt to create disconnected problems that made no sense. To create their problems, students considered special, general, and analogous problems.
- *Reasoning abstractly and quantitatively*. In constructing a dynamic representation of a problem situation, students were engaged in reasoning abstractly and quantitatively as they made sense of the relationships among the different attributes of the problem and translated the verbal representation of the problem into a coherent dynamic diagram.
- *Constructing viable arguments and critiquing the reasoning of others*. In examining their dynamic representation of a problem situation, students generated, justified, and refined mathematical conjectures about the solution of their problems. They communicated their initial problems and conjectures to their peers who often provided suggestions to improve the language and mathematical content of the problems and conjectures. In some cases, students were engaged in both inductive and deductive reasoning to justify their assertions to the possible extent.
- *Modeling with mathematics*. In analyzing the structure of a problem, students applied their mathematical knowledge to create a new problem, mathematize it, and construct a dynamic model to get acquainted with the problem itself and investigate its solution. While drawing a static diagram requires comprehending the structure of a problem, constructing and manipulating a dynamic diagram to model a problem situation requires a deeper understanding of the relationships among the underlying concepts. The use of Dynamic Geometry as a conceptual tool afforded students opportunities not only to make connections between real world situations and mathematical ideas but also to see interconnectedness among different mathematical concepts.

- *Using appropriate tools strategically*. In creating problems and modeling their solutions with Dynamic Geometry, students used this tool strategically to gain insight into the relationships among the attributes of a problem and often into their solution. With the support of Dynamic Geometry, students were able to visualize and better understand the results of varying the attributes of a problem to pose and often solve related problems. Furthermore, students recognized the limitations of the use of technology. They acknowledged that the evidence provided by Dynamic Geometry was empirical and that a mathematical argument is needed to prove and better understand that a plausible conjecture is indeed the solution to the problem. In this respect, the strategic use of Dynamic Geometry furthered students' mathematical thinking.
- *Attending to precision*. By communicating their problems and plausible solutions to others and receiving peer reviews for their creations, students made explicit use of definitions and improved the mathematical language employed in both the problem statement and plausible solution. By modeling the problem situations using the construction, measuring, and dragging capabilities of Dynamic Geometry, students constructed accurate dynamic diagrams that would have not been possible using imprecise paper and pencil constructions.

Reflection and Conclusion

I have always been fascinated and intrigued by the process that leads to the creation of new mathematical problems, at least new to the problem poser. Not having had experience in creating problems throughout the first part of my education, I felt intimated during my first encounter with Brown and Walter's (1990) book, *The Art of Problem Posing*, when I was a doctoral student. Conceptualizing problem posing as an art may detract learners who feel and believe they have no artistic abilities. I believed problem posing to be beyond my reach. As I became familiar with the National Council of Teachers of Mathematics' (1989, 1991) Standards publications, I learned that problem posing had been identified as an important component of students' mathematical experiences. Specifically, the Curriculum and Evaluation Standards for School Mathematics (1989) calls for students to have experiences in formulating their own problems. In a similar vein, the Professional Standards for Teaching Mathematics (1991) advocates that "students should be given opportunities to formulate problems by modifying the conditions of a given problem" (p. 95). Similar statements are included in the most recent Principles and Standards for School Mathematics (NCTM, 2000).

I then became interested in learning more about the process of formulating problems so that I could become a creator of mathematics and provide my students, both prospective elementary and secondary mathematics teachers, experiences in creating problems. The availability of Dynamic Geometry facilitated my conversion from skeptic to evangelist. To help my students learn to pose mathematical problems *systematically* and *spontaneously*, I designed the problem-posing framework displayed in Figure 2. This problem-posing framework has helped me and my students enhance our creativity, at least in the area of posing mathematical problems (Contreras, 2003, 2004, 2007, 2009, in press; Martínez-Cruz & Contreras, 2002).

The problem-posing task described in this classroom experience is one of several problem-posing tasks that I now regularly use with my prospective secondary mathematics teachers when teaching geometry. I should mention, however, that learning to pose interesting and meaningful problems systematically and spontaneously is a challenging process. Problem posers, from naïf to expert, need time to explore the given problem in order to understand the known and unknown attributes of the problem. Exploring the problem through inquiry affords all problem posers and solvers opportunities to better understand the problem. Within geometric contexts, Dynamic Geometry has been a very powerful tool to facilitate the exploration phase. Needless to say, problem posers need to know enough mathematics in order to decide what alternatives or variations to pursue.

Posing and solving mathematical problems is a challenging but worthwhile activity. On the one hand, a lack of proper mathematical background and a failure to explore the problem situation may result in creating problems that are trivial, ill-posed, or uninteresting. On the other hand, subjects may pose problems that are very difficult, if not impossible, for even the instructor to solve. For example, the first version of Problem 5 from Table 2 involved finding a point that minimizes the product of the distances to the sides of the triangle, which is a trivial problem. Then the student considered the point that maximizes such product, another trivial problem without solution. Finally, the point was restricted to the interior of the triangle. We thought that the problem was very interesting and felt joy in posing it, even though the class was not able to solve it. We conjectured that it was the centroid, a surprising result!

Posing and solving problems is adventuring into uncharted territory but the experience is valuable. As a teacher of mathematics, I feel pleasure when students experience a glimpse of *real* mathematics, where problems have neither a simple solution nor a solution that can be found in the back of the textbook. As stated by Ghrist and Lane (2013), "Sometimes the thrill comes not from the final destination, but from the journey" (p. 13).

I encourage everyone: teachers, teacher educators, and instructors at all levels, to provide their students with experiences in posing mathematical problems and to explore posing problems yourself. It is a delight to create and solve new problems, even if they are new only to us.

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