Journal of Mathematics Education at Teachers College

Fall – Winter 2013

A Century of Leadership in Mathematics and its Teaching

TABLE OF CONTENTS

Preface

Articles

TABLE OF CONTENTS

Other

Exploring Mathematical Reasoning of the Order of Operations: Rearranging the Procedural Component PEMDAS

Jae Ki Lee
Susan Licwinko
Nicole Taylor-Buckner
Borough of Manhattan Community College

PEMDAS is a mnemonic device to memorize the order in which to calculate an expression that contains more than one operation. However, students frequently make calculation errors with expressions, which have either multiplication and division or addition and subtraction next to each other. This article explores the mathematical reasoning of the Order of Operations and the effectiveness of a new approach.

Keywords: PEMDAS, order of operation, developmental education, alternative algorithm, case study

Introduction

One of the current trends of teaching in mathematics is the "reduced emphasis on computational skills and more emphasis on developing conceptual learning and deeper understanding" (Westwood, 2011, p. 6). The National Council of Teachers of Mathematics (NCTM, 2000) standards emphasize students' learning methods be based on class activity and teachers' instruction encourage and assist students to develop critical thinking and deeper understanding. However, students are still learning mathematics based on the application of rules, procedures, techniques, and routine practices (Friedlander & Arcavi, 2012).

The Order of Operations is a concept that relies heavily on procedural components in order to solve problems involving more than one calculation. PEMDAS or **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally is the most well-known mnemonic device for remembering the correct order in which operations are to be performed. Students just memorize the procedural component and solve problems without understanding them (Friedlander & Arcavi, 2012; Wu, 2007), and frequently make mistakes whenever any expression includes multiple operations in a random order. Due to this, mathematics textbooks introduce and emphasize another procedure, called the "Left to Right" rule, when simplifying if an expression contains multiplication and division or addition and subtraction next to each other (Akst & Bragg, 2009). Nevertheless, many students still make these mistakes. To motivate students to study mathematics, school provides a solid foundation of basic mathematics (Darley, 2005), and alternative approaches can work well for different types of learners (Lee, Choi & McAninnch, 2012). Until students develop a number sense, it is difficult for them to understand the relationship between different classifications of numbers. In order to develop this number sense, students need to understand the connection among numbers and how to apply number properties and basic operations (Darley, 2005). Additionally, it is important that students explore *why* PEMDAS works, especially the relationship multiplication has with division and the relationship addition has with subtraction, so that they can understand the Order of Operations with mathematical reasoning (Friedlander & Arcavi, 2012). If students can define the Order of Operations with basic mathematics, they can identify the Order of Operations for numerical expressions. This study focused on instructing the Order of Operations emphasizing mathematical reasoning with the hope that students can develop an understanding of why this set of rules are followed. Because of this, the authors suggest the method of "Rearranging Numerical Expressions," which guides students to reorganize a numerical expression.

Related Literature

The Order of Operations developed naturally from the inquiring minds of mathematicians. 'What answer should I get if given $6+2\times3$? What about $8\div2\times2$?' Without an established method, two answers are possible for each expression, 24 or 12 for the former and 8 or 2 for the later. Peterson (2000) found some discussion of precedence with the operations in the 1600s where it was agreed upon that multiplication has precedence over addition. The order issue was sparked by the introduction of exponents in mathematics (Peterson, 2000). Three centuries later, the mathematics historian Florian Cajori noted that the community was in disagreement as to whether multiplication should have precedence over division or if they should be considered equivalent operations (Peterson, 2000). From here, we can presume that the current Order of Operations was decided during this era.

As many researchers agree, memorizing a rule or algorithm means the learner does not have an understanding of the material and is likely to forget it in due time (Lee, 2007; Ohlsson & Rees, 1991). Frequently, when asked to evaluate an expression in which subtraction precedes (i.e., is to the left of) addition, people will add first then subtract, because in the mnemonic device PEMDAS, addition precedes subtraction. For example, given the expression $13-5+6$, many people will arrive at an answer of 2 because $13-(5+6)=13-11=2$. However, the correct answer is found by first subtracting 5 from 13 then adding 6 to the result, giving an answer of 14.

A study by Parsad, Lewis, and Greene (2003) found 22% of first time freshman required at least one remedial mathematics course, while Aldermann (2002) found that 46% of college freshman required remedial mathematics classes. Further, Bahr (2007) found that only 25% of students in remedial mathematics would complete a college-level mathematics course. One reason why these basic skills are so difficult for students to remember could be because they were taught incorrectly the first time they learned it. Glidden (2008) found 30.9% of pre-service elementary school teachers incorrectly added before they subtracted and 38% multiplied before they divided even though both subtraction and division were to the left of the addition and multiplication, respectively, in expressions presented to them.

Rearranging Numerical Expressions

The mnemonic device PEMDAS stands for Parenthesis – Exponents – Multiplication – Division – Addition – Subtraction. Unfortunately, some numerical expressions do not strictly follow PEMDAS. Thus, students should be given an opportunity to explore the Order of Operations. This study generates a way to rearrange expressions, so that students can follow PEMDAS exactly.

Connecting basic mathematics algorithms provides an opportunity for students to understand PEMDAS with mathematical reasoning. Addition and multiplication of expressions are commutative and associative (Aufmann, Barker & Lockwood, 2009; Akst & Bragg, 2009). Frequently, division of fractions is introduced as "Invert and Multiply." Using this technique, $10\div5\cdot2$ can be rewritten as $10\cdot\frac{1}{5}\cdot2$. Since there is now only multiplication, this allows the use of the commutative property: $10\cdot\frac{1}{5}\cdot2\rightarrow10\cdot2\cdot\frac{1}{5}$. From here $10\cdot2\cdot\frac{1}{5}$ is the same as $10\cdot2\div5$ because $2\cdot\frac{1}{5}$ is the same as $2\div5$. More generally, $a\div b\times c$ can be expressed as $a\times c\div b$.

A common way to teach subtraction of integers is to "add the opposite" (e.g., $a+[-b]=a-b$) (Akst & Bragg, 2009, p. 20). For example, the expression $7-5+3$ can be rewritten as $7+(-5)+3$. Using the commutative property it becomes $7+3+(-5)$ and then that is rewritten as $7+3-5$. Again, more generally, $a-b+c=a+c-b$. After rearranging expressions, students can follow the exact PEMDAS order instead of using the "Left to Right" rule with the operations multiplication and division or addition and subtraction. The following example details a rearranged expression from this study.

Example:

$$9\div3\times3\div(6-3)\times5$$

Solution:

$$\begin{aligned}&9\div3\times3\div(6-3)\times5\\&=9\div3\times3\div3\times5\\&=9\times3\times5\div3\div3\\&=27\times5\div3\div3\\&=135\div3\div3\\&=45\div3\\&=15\end{aligned}$$

Methodology

One way to find a significant difference between an alternative approach and a traditional method is to compare two different groups: control and experimental. The control group was instructed with traditional methods and the results were compared with the experimental group, who were instructed using an alternative approach. Several researchers, such as Biesenthal (2006), Bosfield (2004), and Lee (2011), used this method to support their research findings and described the significant difference between experimental and control groups.

Like previous studies, this study also selected two different groups. The control group learned the ordinary method of Order of Operations and the experimental group learned the "Rearranging Numerical Expressions" method to simplify expressions. The research was conducted in a total of three class meetings. Both control and experimental groups were given the same pre-test and post-test to evaluate students' improvement before and after learning the Order of Operations for numerical expressions.

The results were then analyzed using the ANCOVA statistics tool and found the correlation coefficient. In addition, several students' pre-tests and post-tests from the experimental group were selected and examined to observe whether participants actually used "Rearrange Numerical Expressions" to improve their grade.

The pre-test and post-test were created with the approval of the control and experimental group instructors. The pre-tests and post-tests were evaluated from four different professional experts: the investigator, assistant investigator, the instructor of the control group, and the instructor of the experimental group. They then analyzed and agreed upon the difficulty of two exams before giving them to the students.

There were 25 students enrolled in the experimental group. All 25 elected to participate and 22 students completed this study. At the same time, there were 25 students enrolled in the control group with 17 students completing the study. Both groups agreed to participate in the study. Before the beginning of the chapter, all participants took the pre-test, then after the three lessons, they took the post-test.

At the beginning of the session, the experimental group instructor rearranged the order of covering topics because "Rearranging Numerical Expressions" requires students to understand the concept of fractions and negative numbers. The instructor covered fractions and negative numbers before discussing the Order of Operations.

Results

The study compared the correlation of coefficient between pre-test and post-test results of two groups with ANCOVA test results among pre-test and post-test groups in Table 1. Because of the small sample size, the study did not show a statistically significant difference between two groups ($p = 0.247 > 0.05$). Nevertheless, Table 2 shows the experimental group had greater improvement than the control group; on the pre-test, the mean of the control group (1.75) was greater than the mean of the experimental group (1.143).

While on the pre-test more students from the experimental group received a lower grade (Figure 1), after the lessons the experimental group showed greater improvement than the control group. Table 2 describes the differences between two groups. The third column describes the difference between the groups' test scores. The mean of improvement for the experimental group was 1.113 higher than the control group.

Types of Mistakes

We anticipated students would struggle with simplifying numerical expressions with a random order and the pre-test and post-tests are provided for the reader in Figure 2. The pre-test results supported this and showed 95% of both groups did not simplify correctly. After the post-test, 14 of the 22 (64%) participants in the experimental group followed the "Rearranging Numerical Expressions" method and simplified expressions correctly on question #13 (Figure 3) and 11 of the 22 (50%) of participants simplified correctly on question #14 (Figure 4). Based on the statistics of the study, more than 50% of the participants in the experimental group understood the method of "Rearranging Numerical Expressions" and applied it to these two questions.

Conclusion

The acronym PEMDAS is an easy way to remember and simplify any expression containing more than one calculation. However, these mnemonic devices limit students' thinking;

Table 1.

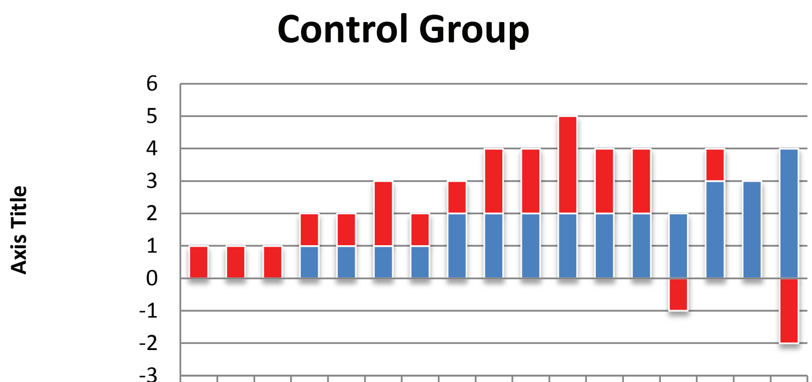
Coefficients[a]

		Unstandardized Coefficients		**Standardized Coefficients**		
Model		B	Std. Error	Beta	t	Sig.
1	(Constant)	1.679	.872		1.925	.062
	Group	.649	.477	.225	1.362	.182
	Pre-test	.229	.193	.196	1.191	2.42

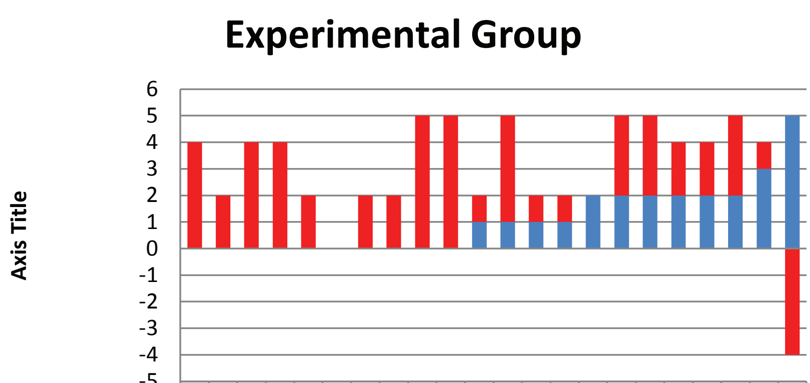
a. Dependent Variable: Post-test

Table 2. Difference Between Experimental and Control Groups on Pre-test, Post-test, and Improvement

Mean	**Control Group**	**Experimental Group**	**Difference (Experiment-Control)**
Pre-test	1.75	1.143	-0.607
Post-test	2.875	3.381	0.506
Improvement	1.125	2.238	1.113



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Improvement	1	1	1	1	1	2	1	1	2	2	3	2	2	-1	1	0	-2
Pretest	0	0	0	1	1	1	1	2	2	2	2	2	2	2	3	3	4



	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
Improvement	4	2	4	4	2	0	2	2	5	5	1	4	1	1	0	3	3	2	2	3	1	-4
Pretest	0	0	0	0	0	0	0	0	0	0	1	1	1	1	2	2	2	2	2	2	3	5

Figure 1. Column Graphs between pre-test and improvement from both groups

PEMDAS Pre-test

1) $4(6-3)^2+8\div 4$

2) $30\div 3+(2+1^8)\times 8$

3) $12-8+1-2+10^3$

4) $40\times 20\div 10\times 2^2$

5) $12\div 4\times 3+6^2$

6) $100-\{6^2+[2(3-1)^3]\div 4\}$

PEMDAS Post-test

1) $3(8-5)^2+12\div 4$

2) $80\div 40+(7+1^4)\times 8$

3) $15-7+3-2+10^2$

4) $30\times 30\div 10\times 3^2$

5) $18\div 6\times 3+4^2$

6) $100-\{8^2+[5(4-2)^3]\div 4\}$

Figure 2. Pre-test and Post-test

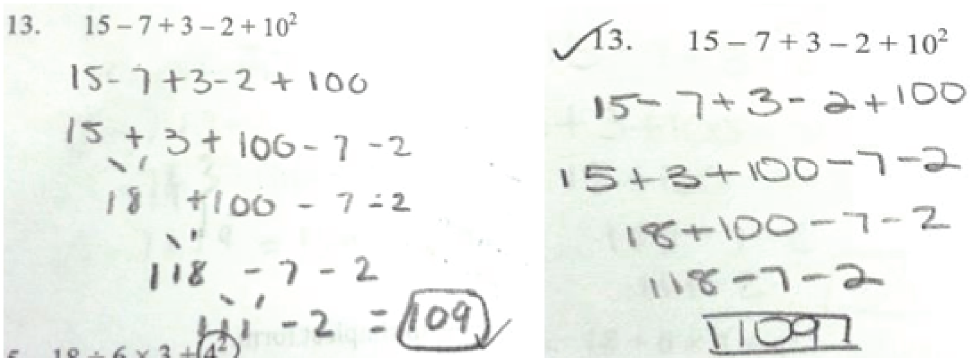


Figure 3. Experimental group sample work – Addition and Subtraction Rearrangement

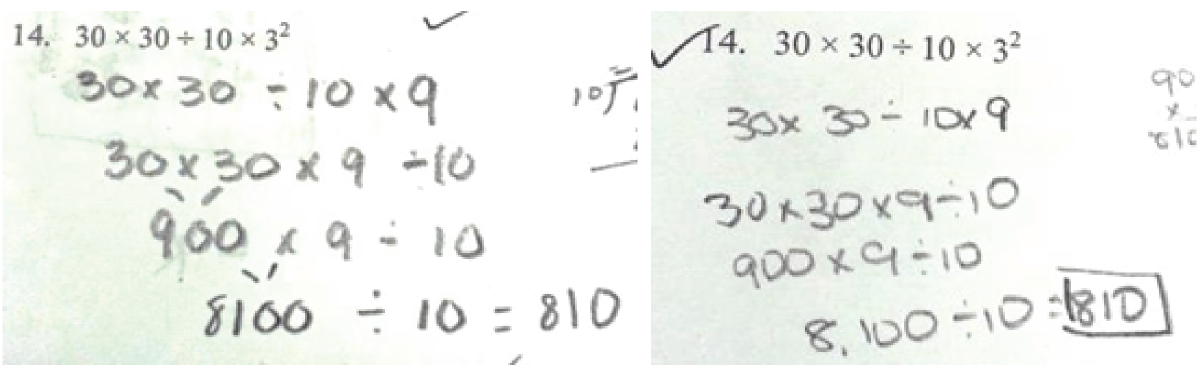


Figure 4. Experimental group sample work – Multiplication and Division Rearrangement

many participants simplify numerical expressions following the mnemonic device without using the "Left to Right" rule necessary for multiplication with division and addition with subtraction. Just memorizing PEMDAS or **P**lease **E**xcuse **M**y **D**ear **A**unt **S**ally will not always provide the correct simplification of a numerical expression. Throughout this study we found that students did not simplify numerical expressions correctly, in both the control and experimental groups, if an expression contained multiple calculations in a random order. It is because students were memorizing rules that they did not understand (Friedlander & Arcavi, 2012; Wu, 2011).

Though we cannot eliminate mnemonic devices to simplify numerical expressions, it is possible to add other factors that guide students to think and find mathematical reasoning to simplify numerical expressions. The alternative approach shown here could be one of the ways to utilize PEMDAS to simplify a numerical expression with mathematical reasoning. If students understand facts such as $a \div b \times c = a \times \frac{1}{b} \times c = a \times c \times \frac{1}{b} = a \times c \div b$ and $a - b + c = a + (-b) + c = a + c + (-b) = a + c - b$, they could rearrange and follow the exact PEMDAS order to simplify a numerical expression.

Most participants in the experimental group understood the "Rearranging Numerical Expressions" method and simplified expressions correctly. Both students and instructors acknowledged that students' mistakes were dramatically reduced after using the "Rearranging Numerical Expressions" approach. Fortunately, the study participants gave more positive feedback than negative feedback. As a few students pointed out, the method sometimes guides students to deal with large numbers because it requires students to calculate all multiplications before any divisions. Further research on the methods tested here, with appropriate adjustments, are necessary to see if the results are well founded.

References

Akst, G. & Bragg, S. (2009). *Intermediate Algebra* (2nd ed.). Boston, MA: Pearson Company.

Aufmann, R., Barker, V. & Lockwood, J. (2009). *Introductory Algebra: An Applied Approach* (7th ed.). Boston, MA & New York, NY: Houghton Mifflin Company.

Bahr, P. R. (2007). Double jeopardy: Testing the effects of multiple basic skill deficiencies on successful remediation. *Research in Higher Education, 48*(6).

Biesenthal, M. (2006). *Students' understanding of fractions within a reform-based instructional program: An action research analysis* (Masters dissertation). Retrieved from ProQuest. (AAT MR24051).

Bosfield, G. F. (2004). *A comparison of traditional mathematical learning and cooperative mathematics learning* (Masters dissertation). Retrieved from ProQuest. (MR1423796).

Darley, J. (2005). *Ninth graders interpretation and use of contextualized models of fractions and algebraic properties: A classroom-based approach.* (Doctoral dissertation). Retrieved from ProQuest. (UMI 3201324).

Friedlander, A. & Arcavi, A. (2012). Practicing algebraic skills: A conceptual approach. *Mathematics Teacher, Vol. 105*(8), 608–614.

Glidden, P. L. (2008). Prospective elementary teachers' understanding of order of operations. *School Science and Mathematics*, *108*(4), 130–136.

Lee, J. (2007). Talking about order of operations. *For the Learning of Mathematics: An International Journal of Mathematics Education, 27*(3).

Lee, J. (2011). *An Asian approach to operations with fractions: Can achievement be enhanced by a mechanistic and heuristic?* (Doctoral dissertation). Retrieved from ProQuest. (UMI 3484371).

Lee, J., Choi, K., & McAninnch M. (2012). An exce-l-ent algorithm for factors and multiples. *Mathematics Teaching in the Middle School, 18*(4), 236–243.

National Council of Teachers of Mathematics (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.

Ohlsson, A. and Rees, E. (1991). The function of conceptual understanding in the learning of arithmetic procedures. *Cognition and Instruction*, *8*(2), 103–179.

Peterson, D. (2000). History of the Order of Operations. Retrieved from http://mathforum.org/library/drmath/view/52582.html.

Wu, H. (2007). "Order of operations" and other oddities in school mathematics. Retrieved from http://math.berkeley.edu/~wu/.

Wu, H. (2011). Phoenix rising: Bringing the Common Core State Mathematics standards to life. *American Educator*, *35*(3), 3–13.