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Fractions, Decimals, and the Common Core

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ABSTRACT At grade 7, Common Core's content standards call for the use of long division to find the decimal representation of a rational number. With an eye to reconciling this requirement with Common Core's call for "a balanced combination of procedure and understanding," a more transparent form of long division is developed. This leads to the formulation of long division as a "recursive relation" and to more incisive insights into fractions and decimals than are typically developed as part of the school curriculum. The role of technology and some modern applications are explored.

KEYWORDS *fractions, decimals, long division, middle school mathematics, Common Core, educational technology*

Introduction

Modern mathematics has been greatly enriched by its reliance on two distinct representations for non-whole real numbers. Accordingly, techniques for converting fractions to decimals and decimals to fractions figure prominently into school curricula. In the *Common Core State Standards for Mathematics*, this is reflected in content standard 7.NS.A.2.D, calling on students at grade 7 to

> Convert a rational number to a decimal using long division; know that the decimal form of a rational number terminates in 0s or eventually repeats. (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA Center & CCSSO], 2010, p. 49)

By way of example, given the rational number 3/7, students are expected to implement the long division algorithm:

$$\begin{array}{r l} & 0.\,4\,2\,8\,5\ldots \\ 7\,) & \overline{3.\,0\,0\,0\,0\ldots} \\ & \underline{0} \\ & \mathbf{3}\,0 \\ & \underline{2\,8} \\ & \quad\mathbf{2}\,0 \\ & \quad\underline{1\,4} \\ & \qquad\mathbf{6}\,0 \text{ etc.} \end{array} \tag{1}$$

and thereby conclude that the decimal form of 3/7 is 0.4285… . Then, reflecting on the nature of these calculations, students are asked to observe that the remainders generated in the course of long division (in bold above) must be less than the divisor (in this case, 7). A remainder of 0 signals that the decimal terminates in zeros. Otherwise, there must eventually be a repetition in the positive remainders so generated, in which case the entire process repeats. In example (1), there is a repeat of the first remainder, 3. This enables one to conclude that $3/7 = 0.\overline{428571}$, where the bar indicates an endlessly repeating pattern of length 6. As such, this seventh grade standard calls for skills that are surely worthy of conveying to students.

Issues Deserving Attention

As would be the case with any comprehensive effort to define the school curriculum, Common Core raises issues deserving attention from teachers and others involved in mathematics education. In the case of the seventh grade standard (7.NS.A.2.D), the fact that long division is cited as *the* tool for converting rational numbers to decimal form can obscure the fact that there are instances where other tools are both available and preferable. Prominent among these is the case of a fraction a/b with denominator of the form $2^m \times 5^n$.

To write 7/250 in decimal form, one can begin by noting that $250 = 2^1 \times 5^3$. Then, multiplying both numerator and denominator by 2^2, one obtains an equivalent fraction whose denominator is a power of 10.

$$\frac{7}{250} = \frac{7}{2^1 \times 5^3} \times \frac{2^2}{2^2} = \frac{7 \times 4}{2^3 \times 5^3} = \frac{28}{10^3} = 0.028$$

In this way, any fraction whose denominator is of the form $2^m \times 5^n$ can be transformed into an equivalent fraction of the form $A/10N$, and thereby into a terminating decimal.

The converse is also true. Given a terminating decimal with N digits to the right of the decimal point, multiplication by $10N$ yields a whole number—say A. This means that the value of the original decimal is a fraction of the form $A/10N$, where $10^N = 2^N \times 5^N$. When reduced to lowest terms, such a fraction has denominator of the form $2^m \times 5^n$.

By characterizing fractions with terminating decimal expansion, we have also characterized those fractions whose decimal expansion does *not* terminate. In particular, a fraction in lowest terms whose denominator has a prime divisor other than 2 or 5 will not be terminating. Rather, as follows from an understanding of long division, its decimal is eventually repeating and has a repetend (repeating pattern) of length less than or equal to $b - 1$. This important observation is expressed in Table 1.

Table 1

Properties of Fractions and Decimals

denominator of proper fraction a/b in the lowest terms	prime factorization is all 2s and 5s	prime factorization not all 2s and 5s
properties of decimal representation of a/b	terminating	repeating

Turning to terminology, the grade 7 Common Core standard asks students to convert a *rational number* to decimal form. Yet our development of the dichotomy underlying Table 1 made essential use of concepts and terminology associated with *fractions* (numerator, denominator, equivalent fractions, etc.). This observation relates to a novel feature of Common Core, namely its early emphasis on a formal interpretation of fractions and their place within the system of real numbers. Reconciling Common Core's content standards for teaching fractions at grades 3 – 6 with subsequent efforts to convey the mathematics underlying Table 1 is an issue worthy of attention.

The seventh grade standard quoted previously is the first of two places where the phrase "long division" appears in Common Core.[1] There is, however, a grade 6 content standard (6.NS.B.2) calling on students to

> Fluently divide multi-digit numbers using the standard algorithm. (NGA Center & CCSSO, 2010, p. 42)

The fact that "standard algorithm" is left undefined raises the question of where, when, and how long division is to be introduced. Should it be taught at grade 6 as a tool for dividing multi-digit numbers? Or should it be introduced at grade 7 as a tool for converting fractions to decimals? Spirited discussion of these matters is contained in Klein and Milgram (2000), Ralston (2000), and Rude (2004).

Practice Standards

The most striking challenge teachers are likely to encounter in implementing these seventh (and sixth?) grade content standards may be one of reconciling them with Common Core's Standards for Mathematical Practice. For many students (and for some teachers!) long division tends to be a mechanistic process: Learning to arrive at 3/7 = 0.4285... by long division does not entail the ability to explain *why* it is that this algorithm generates the decimal representation of a fraction. Yet Common Core's practice standards assert that

[1] The second mention of long division occurs under "High School: Algebra" and deals with division of polynomials.

> The Standards for Mathematical Content are a balanced combination of procedure and understanding... Students who lack understanding of a topic may rely on procedures too heavily... a lack of understanding effectively prevents a student from engaging in the mathematical practices... (NGA Center & CCSSO, 2010, p. 8)

How can teachers help their students arrive at the decimal representations of fractions with “a balanced combination of procedure and understanding?”

An Alternate Format

A transparent way of arriving at the decimal representation of 3/7 is to take advantage of our decimal monetary system. Here the process of dividing \$3 among 7 children can be conceived of as follows:

1. Given \$3 and 7 children, we are unable to give every child a whole dollar.
2. Converting \$3 into 30 dimes, we are able to give 4 dimes to every child and have 2 dimes left over.
3. Noting that we are unable to give every child another dime, we convert the 2 remaining dimes into 20 pennies. Now we are able to give 2 pennies to every child and have 6 pennies left over.
4. Noting that we are unable to give every child another penny, we convert the 6 remaining pennies into 60 milles.[2] Now we are able to give 8 milles to every child and have 6 milles left over.

Having run out of denominations (even the mille may sound unfamiliar), it remains to imagine ever smaller monetary units obtained by successive divisions by ten.

To turn such a verbal description into an algorithmic procedure, we can use Euclidean (whole number) division to give these steps a symbolic representation.

Every child receives 0 dollars	$3 = 0 \times 7 + 3$
Every child receives 4 dimes	$30 = 4 \times 7 + 2$
Every child receives 2 pennies	$20 = 2 \times 7 + 6$
Every child receives 8 milles	$60 = 8 \times 7 + 4$ etc.

Of course, the equations above can be regarded as just another way of implementing the long division algorithm. This can be illustrated by writing the two procedures side by side.

$0.\,4\,2\,8\,5\ldots$	
$7\,)\,3.\,0\,0\,0\,0\ldots$	$\mathbf{3} = 0 \times 7 + \mathbf{3}$
$\underline{0}$	
$\mathbf{3}\,0$	$\mathbf{3}0 = 4 \times 7 + \mathbf{2}$
$\underline{2\,8}$	
$\mathbf{2}\,0$	$\mathbf{2}0 = 2 \times 7 + \mathbf{6}$
$\underline{1\,4}$	
$\mathbf{6}\,0$ etc.	$\mathbf{6}0 = 8 \times 7 + \mathbf{4}$ etc.

Thus, one approach to reconciling Common Core’s seventh grade content standard with a quest for understanding is to use the calculations on the right to explain *why* long division, in its traditional format, leads to the decimal representation of 3/7. In this context, we might simply refer to the calculations on the right as “an alternate format for solving $3 \div 7$” while continuing to teach long division in the usual way.

There may, however, be a case for giving this alternate format a life of its own. Doing so would challenge over 400 years of tradition by bringing “a new way of solving $a \div b$” into the seventh grade curriculum.[3]

A Symbolic Formulation

One rationale for introducing our alternate format for solving $a \div b$ is the development of a mathematical formulation of long division. Closely related to such a development is the use of variables.

Motivated by our now familiar example of $3 \div 7$, a general formulation of long division might be the second set of equations in (2).

$$\begin{array}{lrl} & 3 = 0 \times 7 + 3 & a = q_0 \times b + r_0 \\ & 30 = 4 \times 7 + 2 & 10r_0 = q_1 \times b + r_1 \\ & 20 = 2 \times 7 + 6 & 10r_1 = q_2 \times b + r_2 \\ (2) & 60 = 8 \times 7 + 4 & 10r_2 = q_3 \times b + r_3 \text{ etc.} \end{array}$$

The problem of writing a/b in decimal form then becomes one of solving a *recursive relation*

$$\begin{array}{ll} & a = q_0 \times b + r_0 \\ (3) & 10r_{n-1} = q_n \times b + r_n \text{ for } n = 1, 2, \ldots \end{array}$$

and using the sequence of quotients $\{q_0, q_1, q_2, q_3, \ldots\}$ to arrive at

$$(4) \qquad a/b = q_0.q_1q_2q_3\ldots$$

[2] The Coinage Act of 1794 defines a mille as "the thousandth part of a dollar."

[3] The mathematician Henry Briggs is credited with developing long division around 1600 (*Dictionary of Scientists*, 1999).

Such an abstract formulation makes clear the rather specialized role long division plays in finding the decimal representation of fractions. Here, the traditional form of long division calls for "bringing down zeros," which is different from its use to "fluently divide multi-digit numbers." At grade 6, students are confronted with a multi-digit dividend $a = a_n a_{n-1} \ldots a_1 a_0$ and the use of long division to solve $a \div b$ becomes one of finding quotients $q_n, q_{n-1}, \ldots, q_0$ and remainders $r_n, r_{n-1}, \ldots, r_0$ satisfying

$$a_n = q_n \times b + r_n$$

$$(5) \quad 10r_k + a_{k-1} = q_{k-1} \times b + r_{k-1} \text{ for } k = n, n-1, \ldots, 1$$

The solution of $a \div b$ is now the whole number

$$(6) \quad q_n, q_{n-1}, \ldots, q_0$$

accompanied by the remainder r_0. As the general formulations (3) and (5) make clear, long division is more complicated at grade 6 than it is at grade 7!

Technology

In pursuing the relationship between fractions and decimals, the existence of computer technology is hard to ignore. There will surely be instances in which it becomes tempting to relegate the decimal representation of fractions to a machine, and a recursive definition of long division is likely to be central to any such effort.

An engaging way of pursuing recursion in a non-technical context is to ask students to implement the alternate format of long division on a spreadsheet. Given a rudimentary background in Excel, students can be given a template such as (A) at the top of Figure 1 and asked to "teach the machine" the monetary scheme for dividing *a* dollars among *b* children.

For large values of *a* and *b*, such a spreadsheet can be used to calculate the decimal form of *a*/*b* to an arbitrary number of decimal places. In this way, technology provides a powerful tool for experimentation and for thinking independently about the topic at hand.

In a real world context, it also seems important to lay the groundwork for a mathematical understanding of the bar codes, parity checks, and forms of credit card encryption that play an increasingly important role in our society. Remarkably, some of these applications are closely related to the mathematics of fractions and decimals. By way of dramatic example, an understanding of fractions, decimals, and long division can be used to establish Fermat's Little Theorem. As recounted by Flannery and Flannery (2001) in a book aimed at students, this 350 year old result played a central role in the development of a form of "public key encryption" called the RSA algorithm. As such, it will be interesting to consider its roots in the school curriculum.

	A	B	C	D	E	F
1	This spreadsheet allows you to assign a and b. It converts a/b to decimal form by implementing the formula $10r_{n-1} = b \times q_n + r_n$ for n = 0, 1, 2, ... and then summing the terms $q_n \times 10^n$.					
2						
3	a=	3	b=	7	a/b =	
4						
5	n	$10r_{n-1}$	b	q_n	r_n	Solver
6	0	3				
7	1					
8	2					
9	3					
10	4					

(A)

	A	B	C	D	E	F
1	This spreadsheet allows you to assign a and b. It converts a/b to decimal form by implementing the formula $10r_{n-1} = b \times q_n + r_n$ for n = 0, 1, 2, ... and then summing the terms $q_n \times 10^{-n}$.					
2						
3	a=	3	b=	7	a/b =	0.4285
4						
5	n	$10r_{n-1}$	b	q_n	r_n	$q_n \times 10^{-n}$
6	0	3	7	0	3	0
7	1	30	7	4	2	0.4
8	2	20	7	2	6	0.02
9	3	60	7	8	4	0.008
10	4	40	7	5	5	0.0005

(B)

Figure 1. Using a spreadsheet to implement recursion.

Decimals to Fractions

Having considered techniques for going from the fraction representation of a rational number to its decimal representation, there remains the problem of transforming repeating decimals into fractions. Recalling the familiar example 1/3 = 0.3333…, it remains to show how the decimal 0.3333... leads to 1/3.

Letting x = 0.3333…, a rigorous effort to arrive at x = 1/3 would involve the interpretation of 0.3333... as a geometric series with limit 1/3. There is, however, a less formal procedure that is essentially equivalent:

Decimal notation and basic algebra
can be used to argue that

$$\begin{array}{r} 10x = 3.3333... \\ \underline{x = 0.3333...} \\ 9x = 3.0000... \end{array}$$

Solving the last equation, we arrive at
$x = 3/9$ or $x = 1/3$.

These ideas are readily applied to decimals with longer repetends *as long as the repetend begins right after the decimal point*. For example, given $x = 0.454545\ldots = 0.\overline{45}$ we have

$$\begin{array}{r} 100x = 45.454545... \\ \underline{x = \ \ 0.454545...} \\ 99x = 45 \end{array}$$

and $x = 45/99 = 5/11$.

In order to deal with repeating decimals having a "lead-in" preceding the first repetend, we can reduce the problem to the case considered above. Given $x = 0.17454545\ldots = 0.17\overline{45}$, we have

$$\begin{aligned} x &= 0.17454545... \\ 100x &= 17.454545... \\ 100x &= 17 + 0.454545... = 17 + 5/11 = 192/11 \\ x &= 192/1100 = 48/275 \end{aligned}$$

These techniques make it possible to transform any endlessly repeating decimal into fraction form. This leads to a characterization of *non*-repeating decimals as numbers that can *not* be represented in fraction form (aka irrational numbers).

A Refined Dichotomy

As illustrated previously, when transforming repeating decimals into fraction form, it is useful to distinguish between decimals with a repetend beginning right after the decimal point versus those having a "lead-in" before the first repeating pattern. In refining the dichotomy of Table 1, the following questions arise:

> Given a fraction a/b in lowest terms, when will the decimal representation of a/b have a repetend right after the decimal point? When will its decimal representation have a lead-in before falling into a repeating pattern?

As asserted in Table 2, the answer is again rooted in the prime factorization of the denominator b.

While not particularly difficult (see Appendix), a proof of this refinement of Table 1 may be beyond the school curriculum. Notable, however, is its reliance on the recursive formulation of long division as a starting point.

$$\begin{aligned} a &= q_0 \times b + r_0 \\ 10r_{n-1} &= q_n \times b + r_n \text{ for } n = 1, 2,\ldots \end{aligned}$$

By way of contrast, the traditional format for long division would be an awkward basis for establishing Table 2.

Short of proof, it is still possible to relate such a refinement of Table 1 to the school curriculum. For example, a spreadsheet can be used for calculations aimed at confirming Table 2 in specific cases. Alternatively, experimentation can be used as a way of guiding students to the "discovery" of these results. So motivated, students in Math Circles and similar programs of enrichment can be referred to Chapter 23 of *The Enjoyment of Mathematics: Selections from Mathematics for the Amateur* by Rademacher and Toeplitz (1957) as a context for continued study.

Given Table 2, it becomes possible to develop some engaging connections to the standard curriculum. In confronting the seventh grade Common Core content standard cited at the beginning of this discussion, students will note that some fractions have decimal representations with surprisingly short repeating patterns

$$7/11 = 0.\overline{63} \qquad 5/37 = 0.\overline{135}$$

while others have unpleasantly long repetends.

$$3/7 = 0.\overline{428571} \qquad 12/17 = 0.\overline{7058823529411764}$$

Table 2

Properties of Fractions and Decimals Refined

denominator of proper fraction a/b in lowest terms	prime factorization is all 2s and 5s	prime factorization has no 2s and 5s	prime factorization has some 2s and/or 5s and also some other primes
properties of decimal representation of a/b	terminating	repetend right after decimal point	first repetend preceded by lead-in terms

This raises the question:

> Given a fraction a/b in lowest terms, what is it that determines the length of the shortest repetend in its decimal expansion?

To answer to this question, consider a fraction a/b whose decimal representation has a repetend of length L right after the decimal point. Setting $x = a/b$ and

$$x = 0.c_1c_2c_3\ldots c_L c_1c_2c_3\ldots c_L c_1c_2c_3\ldots c_L\ldots = 0.\overline{c_1c_2c_3\ldots c_L}$$

we have

$$10^L x = c_1c_2c_3\ldots c_L.c_1c_2c_3\ldots c_L c_1c_2c_3\ldots c_L\ldots$$

and

$$(10^L - 1)x = c_1c_2c_3\ldots c_L$$

Solving for x in fraction form, we arrive at

$$x = \frac{c_1c_2c_3\ldots c_L}{10^L - 1} = \frac{N}{10^L - 1}$$

where N is the whole number $c_1c_2c_3\ldots c_L$. Since $x = a/b$,

$$\frac{a}{b} = \frac{N}{10^L - 1}$$

This proportion leads to

$$b \times N = a \times (10L - 1)$$

and shows that b is a divisor of $a \times (10L - 1)$. But since a/b is in lowest terms, we can also conclude that b is a divisor of $10L - 1$. Then, after working our way back through these calculations, we arrive at the following consequence of Table 2.

> Given a fraction a/b in lowest terms whose denominator is divisible by neither 2 nor 5, the decimal representation of a/b will begin with a repetend of length L if and only if b is a divisor of $10L - 1$.

So what are some fractions whose decimal representation consists of a repetend of length L beginning right after the decimal point? Here are some lists of their denominators for small values of L:

$L = 1$	$(10^1 - 1) = 9$	$b = 1, 3, 9$
$L = 2$	$(10^2 - 1) = 99$	$b = 1, 3, 9, 11, 33, 99$
$L = 3$	$(10^3 - 1) = 999$	$b = 1, 3, 9, 27, 37, 111, 333, 999$
$L = 4$	$(10^4 - 1) = 9999$	$b = 1, 3, 9, 11, 33, 99, 101, 303,\ldots$
$L = 5$	$(10^5 - 1) = 99999$	$b = 1, 3, 9, 41, 123, 271, 813,\ldots$
$L = 6$	$(10^6 - 1) = 999999$	$b = 1, 3, 7, 9, 13,\ldots, 3367,\ldots, 5291,\ldots$

and here are some examples that make use of these lists:

$L = 1$	$1/3 = 0.3333\ldots$	$7/9 = 0.6666\ldots$
$L = 2$	$5/11 = 0.454545\ldots = 0.\overline{45}$	$2/9 = 0.2222\ldots = 0.\overline{22}$
$L = 3$	$8/37 = 0.\overline{216}$	$2/9 = 0.2222\ldots = 0.\overline{222}$
$L = 4$	$5/11 = 0.\overline{4545}$	$6/101 = 0.\overline{0594}$
$L = 5$	$1/41 = 0.\overline{02439}$	$10/41 = 0.\overline{24390}$
$L = 6$	$3/7 = 0.\overline{428571}$	$617/5291 = 0.\overline{116613}$

Conclusion

Efforts to reformulate school curricula can be trying for both schools and teachers. At the same time, they offer unique opportunities to examine practices that are deeply rooted in tradition and habits of thought.

By examining long division in light of Common Core's Standards for Mathematical Practice, we have been led to alternate ways of viewing an algorithm that is likely to be a challenging part of any curriculum. But the value of such alternative approaches will not be deduced in abstract terms. Rather, such a determination will require the active involvement of classroom teachers willing to dig into the underlying mathematics and try new approaches in their work with students. Hopefully, our discussion of fractions, decimals, and the Common Core will contribute to such efforts.

Appendix

Theorem If a/b is a proper fraction in lowest terms whose denominator is divisible by neither 2 nor 5, the decimal representation of a/b has a repetend right after the decimal point.

Proof. Letting $0.c_1c_2c_3\ldots$ denote the decimal representation of a proper fraction a/b in lowest terms, suppose that $c_mc_{m+1}\ldots c_n$ is the first repetend to the right of the decimal point. In applying the recursive form of long division algorithm to $a \div b$, we obtain

$$\begin{aligned} a &= 0 \times b + r_0 \\ 10r_0 &= c_1 \times b + r_1 \\ 10r_1 &= c_2 \times b + r_2 \\ &\ldots \\ 10r_{m-1} &= c_m \times b + r_m \\ &\ldots \\ 10r_{n-1} &= c_n \times b + n \end{aligned}$$

with $r_m = r_n$. Subtracting the last two equations displayed, we obtain

$$10(r_{n-1} - r_{m-1}) = (c_n - c_m) \times b$$

Since $10 = 2 \times 5$ and *b* is divisible by neither 2 nor 5, it follows that *b* is a divisor of the difference $r_{n-1} - r_{m-1}$. But recalling that

$$0 < r_{n-1} < b \text{ and } 0 < r_{m-1} < b$$

the only way for *b* to divide $r_{n-1} - r_{m-1}$ is for this difference to be 0. Thus the assumption $r_m = r_n$ has led to the conclusion $r_{n-1} = r_{m-1}$ and thereby to $r_k = r_0$ for some *k*. This shows that the decimal representation of *a*/*b* has a repetend right after the decimal point.

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