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Striking a Balance: Students' Tendencies to Oversimplify or Overcomplicate in Mathematical Modeling

Heather Gould
Eugene Lang College, The New School

Nicholas H. Wasserman
Teachers College Columbia University

ABSTRACT With the adoption of the *Common Core State Standards for Mathematics* (CCSSM), the process of mathematical modeling has been given increased attention in mathematics education. This article reports on a study intended to inform the implementation of modeling in classroom contexts by examining students' interactions with the process of creating mathematical models and analyzing the types of models so generated. Results indicate students' tendencies to oversimplify or overcomplicate the modeling process. Implications of the study are discussed, especially for understanding which aspects of the modeling cycle might be most helpful for teachers to focus on in order to develop students' modeling abilities.

KEYWORDS *mathematical modeling, student modeling process, mathematization, middle school mathematics, algebra, Common Core*

Recently, mathematical modeling has garnered increased national attention in education, partly stemming from its inclusion in the *Common Core State Standards for Mathematic*s (CCSSM). The mathematical modeling cycle includes several important features that distinguish modeling from mathematical activities typically found in the classroom. In particular, modeling requires students to perform three distinctive processes: make assumptions about a real-world situation in order to identify the mathematical "ground rules" for the situation; mathematize a real-world situation in order to translate it into mathematical terms; and revise an initial solution in order to fix problems with the model or to refine it further. It is not surprising, then, that students frequently encounter difficulties with these parts of the modeling process in particular. As a result of a study looking at how students engage with a mathematical modeling problem, we found that, in trying to adjust to modeling activities, students' work often resulted in one of two extremes: overly-simplified or overly-complicated mathematical models. In this paper, we describe the study and results, as well as discuss implications from the findings.

Mathematical Modeling in the CCSSM

The emphasis on mathematical modeling is simultaneously new and not new. In the widely influential *Principles and Standards for School Mathematics* (2000), for example, the National Council for Teachers of Mathematics (NCTM) describes the need for students to understand mathematical models as mathematical representations in an idealized setting. Indeed, they describe and define models, but emphasize problem solving, rather than mathematical modeling, as a primary process in mathematics education. Note that while problem solving and modeling are related processes, there are fundamental differences (e.g., Lesh & Yoon, 2007; Pollak, 2003). By mentioning "models," but not explicitly addressing the modeling process, the *Principles and Standards for School Mathematics* potentially leave room for various interpretations of how to incorporate models into mathematics education. For example, students could engage in *model analyzing*, which would involve providing students with a specific model and asking them to understand or interpret it; *model recreating*, which would involve having students recreate a specific, known mathematical model; or *model creating*, which would involve having students engage in modeling to create their own mathematical models for a situation. The latter is most akin to having students engage in the process of mathematical modeling, which is the type of task described in this paper.

Some argue that the process of mathematical modeling, perhaps more than anything else, helps achieve the aims of mathematics education (e.g., Blum, 1991). Galbraith (2007) distinguishes two reasons to include modeling in mathematics education: "modeling as vehicle," in which the purpose of modeling is to learn about other mathematical ideas and concepts, and "modeling as content," in which the purpose of modeling is learning to engage with the real world using mathematics. Although the CCSSM (National Governors Association Center for Best Practices, Council of Chief State School Officers [NGA Center & CCSSO], 2010) may emphasize one way slightly more than the other, both views are present (e.g., Gould, 2013). Indeed, in contrast to the *Principles and Standards for School Mathematics*, the CCSSM standards explicitly articulate modeling as a process—one in which students should engage in their mathematical education. Within the CCSSM, mathematical modeling is both a "Standard for Mathematical Practice" for all grades and also a required conceptual category, much like algebra or statistics, at the high school level (NGA Center & CCSSO, 2010). The modeling cycle in the CCSSM is described as having six steps:

(1) identifying variables in the situation and selecting those that represent essential features,

(2) formulating a model by creating and selecting geometric, graphical, tabular, algebraic, or statistical representations that describe relationships between variables

(3) analyzing and performing operations on these relationships to draw conclusions,

(4) interpreting the results of the mathematics in terms of the original situation,

(5) validating the conclusions by comparing them with the situation, and then either improving the model or, if it is acceptable,

(6) reporting on the conclusions and the reasoning behind them. (pp. 72–73)

Making assumptions occurs in step (1), mathematization occurs in step (2), and revision and refinement occur in step (5). Steps (3), (4), and (6) are usually familiar to teachers and students, as they are reminiscent of some of the more familiar aspects of problem solving. Indeed, in the implementation of mathematical modeling, even teachers are frequently unfamiliar with and have misconceptions about the process (e.g., Doerr, 2007; Gould, 2013). Thus, in learning to model with mathematics, and learning to teach to model with mathematics, it can be argued that a significant focus needs to be placed on the acquisition of the skills that are specific to the process of mathematical modeling: those skills required to make assumptions, to mathematize, and to revise or refine. Indeed, there has been less focus on understanding the implementation of modeling in mathematics education (Lingefjärd, 2007), about which this study, by looking at students' interactions with the process of model creating, aims to inform further.

Methods

Based on a limited understanding in the literature about how students work with problems that require them to engage fully in the mathematical modeling process, we sought to address this need. Given that in the United States mathematical modeling is less familiar than problem solving, we were particularly interested in the ways in which students engage with the modeling-specific skills of making assumptions, mathematizing a real-world situation, and revising or refining models based on testing its utility in the original, real-world context. We focused specifically on students' abilities to recognize and choose from among the most important assumptions and variables. Given their unfamiliarity with modeling, students were given a brief introduction to the topic to familiarize them with the process before being asked to embark on a modeling task of their own.

Participants

Eighteen groups (n=18) of 3-4 students each from a large private secondary school completed a mathematical modeling problem. Most of the students were in an advanced mathematics track, coming from seventh, eighth, and ninth grade mathematics classes in algebra and geometry. The students were given a brief (~10 minutes) introduction to mathematical modeling, using one example (involving triangulating in order to estimate temperature at a given point—see Gould, 2012). Students, guided by the teacher, discussed three different mathematical models for the problem, each with increasing complexity, and each in response to limitations and assumptions associated with the models. The groups were then given the remainder of the class period, approximately 45 minutes, to work on creating a mathematical model for one problem. Their work was recorded using SmartPens, which captured both the written work as well as the audio from each group. Small groups were used as a way to compel students to discuss their thoughts and ideas aloud, which resulted in the ability to capture and analyze their rich discussions aligned to specific moments of their work on the problem. Both the final models, captured on paper, and their processes, captured in their discussions, were analyzed to understand how students engaged in the process of modeling. In this initial analysis of the data, the researchers used the constant comparative method of a grounded theory approach (e.g., Strauss & Corbin, 1990) to develop meaningful themes across the corpus of data from the different groups' work, constantly testing and refining the assertions to make sure they were representative of the evidence leading to them.

The Gas Station Problem

The mathematical modeling problem used in this study, "The Gas Station Problem," adapted from resources from the Consortium for Mathematics and its Applications (COMAP, 2006), poses a predicament that is very common among drivers: Is the difference in the price of gasoline at a further station worth the trip, and, among several stations, which is the best from which to purchase gasoline? This problem, along with a table that describes prices at different gas stations and the distances to those stations (see Table 1), was given to the 18 groups of students. Students were instructed to create a mathematical model to determine which gas station to use and to list the assumptions they made explicitly; they were given the option to consider other important variables in the problem besides distance and price.

Table 1

Gas Station Problem Distances and Prices

Gas Station	Distance	Price	*Any other categories you have used in your model*			
Current	0	$3.31				
A	1.1 miles	$3.28				
B	4.5 miles	$3.23				
C	3.1 miles	$3.25				
D	0.5 miles	$3.30				
E	12.0 miles	$3.11				
F	2.2 miles	$3.27				
G	8.7 miles	$3.20				

Results

In most models, students made use of the distance and price of gasoline (but, as is typical with mathematical modeling problems, often in unexpected ways!). Many students also made use of the time it would take to travel to each individual station and considered this as an important variable in the model. Most groups also recognized that important variables include the amount of gasoline needed to fill the tank and the car's fuel efficiency. Groups concerned with the amount of time to travel to each gas station also usually considered the speed of travel. Indeed, the students made various assumptions about these different variables, as well as many others. Upon further analysis aimed at answering the primary question of the study, we found that their resultant models and their engagement with the modeling process, however, often fell into a category of either oversimplification or overcomplication.

Oversimplification

Oversimplification in the gas station problem frequently occurred for one of two reasons. The first reason students tended to oversimplify was a fundamental misunderstanding of the real-world situation. The second was the inability to choose the most important variables and assumptions among the many variables and assumptions that could have been considered, which resulted in frustration and, in turn, a model that failed to address *any* of the important variables and assumptions.

Misunderstanding the real-world situation and unrealistic assumptions. Some students had difficulty interpreting the purpose of the modeling problem, understanding the real-world situation describing the modeling situation, or determining what types of assumptions were realistic to make. As a result, oversimplified models were created. One group, for example, understood that the purpose was to save money in filling the tank and even understood the situation as far as to state, "[the] money you use to drive there has to be less than the money you're saving," but they did not understand the actual situation and the assumptions they made about the situation were quite unrealistic. For example, some members of this group understood the real-world situation to be that the driver fills the tank first at the "current" station, then drives to station A, fills the tank again, drives to station B, fills the tank again, and so on. This is decidedly not a real-world situation. These students failed to understand the reality that the driver has some control over where to fill the tank and would only fill it when necessary. Furthermore, these students were unable—or in some cases unwilling—to make reasonable assumptions regarding fuel efficiency or the size of a gas tank. They decided that the tank could hold only 1 gallon of gas. They also did not understand the meaning of "filling" a gas tank: they evidently believed that the tank must be completely empty before it could be filled. As one student stated, "I thought our tank was only one gallon; we didn't use a whole gallon, so we can't fill up one gallon." The other students agreed, which led them to arrive at an unrealistic conclusion: the tank starts full with 1 gallon of gas, the driver goes to the gas station in question, dumps the remaining gas out, and refills the tank (Figure 1). Seemingly, this group was unable to connect to the real-life situation or determine "how much" of the gas would actually be used in driving to the station, effectively simplifying the problem by dumping whatever remained.

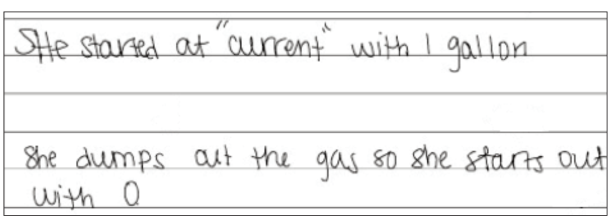

Figure 1. Oversimplified assumption based on misunderstanding of the real-world situation.

Not surprisingly, this lack of understanding of the real-world scenario and inability to make reasonable assumptions led to an inappropriate model. These students indicated a degree of frustration with the problem and resorted to making some assumptions simply for the purpose of easing their workload.

Inability to choose the most important variables and assumptions. Some students were unable to interpret the essence of the problem, which caused them to oversimplify the situation. In one group's initial model, relevant factors included the time to get to each station (a factor that assumed constant speed at 60 miles per hour for convenience of calculation and traffic that did not inhibit transit time) and the volume of the car's gas tank, which was assumed to be 20 gallons. The price of a gallon of gas at each station was also implied as a relevant factor. However, the gas consumed in getting from one station to the next was not taken into account. Instead, the group assumed that the amount of gas to fill the tank was consistent at 20 gallons per tank. Indeed, assuming that they are to fill up 20 gallons at each station will result in simply selecting the cheapest gasoline, effectively missing one of the hallmark aspects of the problem. Figure 2 shows the group's calculations for the price of filling up 20 gallons and the time it would take to travel to each station.

In addition, this group was unable to account for time spent in a meaningful, mathematical way in the initial model, which resulted in further oversimplification of the model. From their calculations, the students decided that stations F and C gave the best balance of a lesser price than, and a reasonable travel time from, the "current" station. (Note that due to a problem with organizing the model on paper, the group incorrectly decided that station F was better than station C "if you can make it," however, this is not a concern with their ability to model, but their ability to organize.) This model and decision was based partially on mathematics—the price of a 20 gallon fill-up—and in part on other more general impressions—about how much time is used in travelling to the next-best station. Had the group decided to account for the gas used in getting to the station and the car's fuel efficiency, the results may have been different.

Encouragingly, this group eventually did realize that the amount of gas used to travel to each station and the car's fuel efficiency were relevant to the problem. They made some attempts to identify how much more gas was used in traveling to more distant stations, but they were unable to complete their task and create a new, revised model accounting for this important factor, effectively leaving the model as it was. Overall, their engagement with the process of modeling was characterized by a tendency to oversimplify, particularly out of a confused understanding of (and, later, an inability to deal with) the most important aspects of the situation.

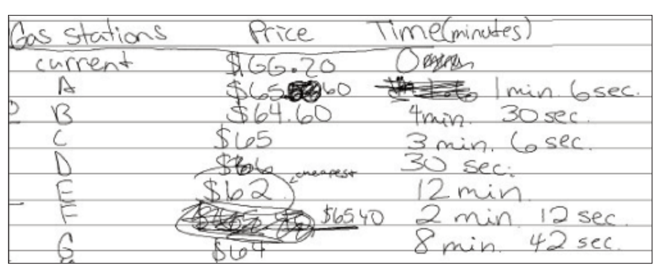

Figure 2. Oversimplified initial model that does not consider gas used in transit to station.

Overcomplication

Overcomplication in the gas station problem occurred because students were, again, not able to choose the most important variables and assumptions among the ones that might be considered. In this case, students incorporated too many of their variables and assumptions into their models at the cost of creating a manageable model.

One group was able to make many of the most important assumptions in the gas station model: the fuel efficiency between stations was assumed to be equivalent (e.g. the traffic conditions were the same along all roads) and was set at 20 miles per gallon, the gas tank was assumed to hold 20 gallons, and the amount of gas left in the tank was sufficient to travel from the "current" station to any other station. An important assumption was overlooked (once again, due to a misunderstanding of the real-world situation), which is that a driver typically knows approximately how much gas will be needed to fill a tank. The students did not understand this. The result was a mathematically correct but very complicated model in which pairwise comparisons were made between "current" and every other station. This model resulted in what the group called "decision points," which were equilibrium points about the quantity of gas that would need to be purchased for the total prices at the two stations to be equal. They used these comparisons to conclude if the driver needed more than, say, 8.5 gallons of gas, then the driver should go to the further station, but should otherwise stay at the current station. Figure 3 shows one of the pairwise comparisons this group made, along with the decision point statement. The students were unable to make all the comparisons necessary to determine the best station at which to fill the tank, as they would have needed to make 28 different comparisons. Even if those comparisons were made, the model created also lacked a definitive method of choosing a station because the students were unable to compare the decision points to one another. Ultimately, they chose the best station to go to based on the idea that a driver filling a tank would fill it with at least 9 gallons of gas. However, once left with the stations at which the decision point produced a need to fill at least 9 gallons to make the trip worthwhile, they made an essentially arbitrary choice of which station to utilize. Without understanding that some known amount of gas was needed initially, the resultant model was mathematically sound, but in reality, too cumbersome to be useful.

"Just Right"

Modelers desire to incorporate the many variables and assumptions they consider to be important but also recognizes the need to develop an initial, manageable model first in order to understand the real-world situation better. The very few groups that were able to strike a balance between this desire and this need did so by employing revision and refinement in the mathematical modeling process.

The real-world situation surrounding this model is, admittedly, quite complex. One group understood this and listed many concepts to consider as possible variables and assumptions: gas mileage (17 miles per gallon), volume of gas tank and the amount of gas needed initially (30 and 25 gallons, respectively), traffic conditions (speed of travel was assumed to be consistent at 40 miles per hour, including a scheme for determining the numbers of red and green lights along the way along and time waited at red lights), and travel time (a function of speed, distance, and the number of red lights encountered, with a maximum of 30 minutes allowed per round trip to get gas). Various other assumptions were initially considered. This group stands out as having created a "just right" model because these students recognized that not all variables and assumptions could or should be included initially or all at once. This group made an initial model that accounted only for the cost of gas needed at the start and the cost of gas that was used to travel to more distant stations. See Figure 4 for part of the formula used in the initial model.

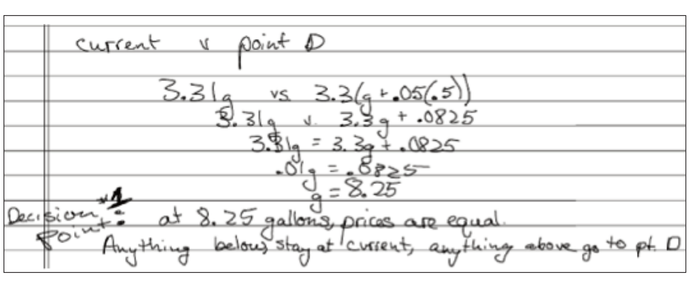

Figure 3. Pairwise comparison of "Current" and station D, along with "Decision Point."

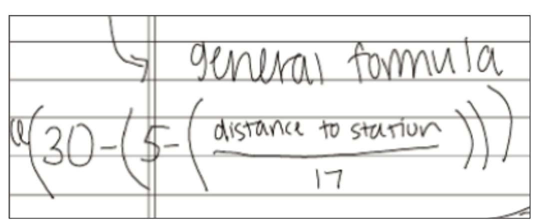

Figure 4. General formula model for determining amount of gas to fill up at each station, to be multiplied by price per gallon.

Once the initial model was completed, the students were able to see that while there was a single, most cost-effective station at which to fill the tank (station E, based on their assumptions), it was not necessarily the best station at which to fill because they were willing to sacrifice some money saved in order to make it home within a reasonable amount of time. Having allowed themselves to go 30 minutes out of the way for a round trip, they determined the travel time if all traffic lights are assumed to be green, then the travel time if all traffic lights are assumed to be red, leaving 1 minute of stop time for every red light, with traffic lights every half mile. It was also assumed that the gas used waiting at red lights was negligible. Using the model incorporating all red lights, the students eliminated possible stations in order from cheapest to most expensive as they found they did not have enough time to travel to the cheapest of the stations in the worst case scenario. This resulted in the "most efficient" choice, which was the cheapest station accessible in the allotted amount of time, station C. This model incorporated all the most important features of the real-world situation while remaining manageable and useful. Figure 5 shows the tables used to organize the final models.

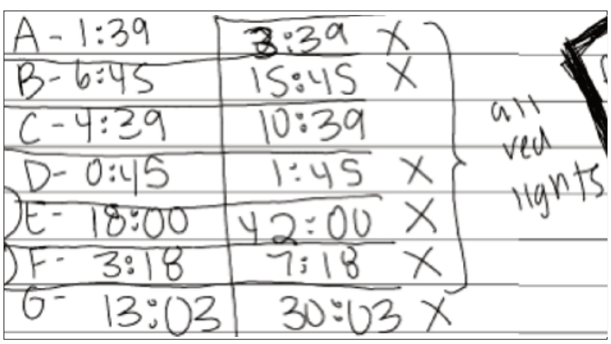

Figure 5. Models for all green lights (left column) and all red lights (right column).

Implications for the Teaching of Mathematical Modeling

Given the emphasis on mathematical modeling present in the widely-adopted CCSSM, teachers will need to find meaningful ways to incorporate modeling tasks into the curriculum. Indeed, modeling tasks can vary, for example, from understanding a known mathematical model to creating a new mathematical model. The problem given in this study is an example of the latter, and is particularly related to the emphasis on modeling as a process in which students should engage: students were asked to create their own mathematical model for a problem. Modeling involves keeping various tensions in balance, particularly between the real-world situation and the mathematical model that intends to give meaningful information about it. Evident from this study, students' engagement with model-creating frequently resulted in models that are either over-simplified or overcomplicated.

Even with only a brief introduction to modeling, students seemed to have little difficulty listing some of their assumptions about the problem; however, understanding that students also tend to have difficulty identifying the right balance between assumptions and mathematics can be useful information for teachers. In particular, students frequently had difficulty figuring out the most important features and necessary assumptions represented in the real-world context. (This is discussed more in the following paragraph.) For some students, modeling tasks may be viewed as an "escape" from having to do any mathematics—just simplify the model "to the nth degree," state the requisite assumptions, and the model is okay. Indeed, nearly every modeling problem could have such a solution. The key for teachers in implementing modeling tasks for these students will come down to an ability to problematize their oversimplified models by pointing out their large gap from reality, and motivating further engagement in the problem to improve the model. This may require pointing students toward a better understanding of the actual real-world situation and its salient aspects. Other students may become overwhelmed by trying to use and incorporate all of the assumptions, ultimately resulting in a model that is not manageable. Such students may become frustrated with a model that is too cumbersome, or ultimately end up making somewhat arbitrary choices to get around the complexity of their assumptions and the mathematics. For teachers, helping such students let go—even temporarily—of some of their reasonable assumptions in order to generate a model that is actually meaningful and manageable will be one key for productive implementation of modeling tasks. Evident from the "just right" example, in order to create a productive model, modelers must balance the desire to incorporate important features with the need to work within reasonable limits by recognizing that simpler initial models should ultimately lead to more robust models through the processes of revision and refinement.

In addition, given the prominence of the "real world" component of modeling, the process of modeling requires thorough familiarity with the specific context. In this regard, perhaps more so than in other arenas, the use of mathematical modeling in school contexts likely needs to draw on real-world contexts with which the specific group of students are very comfortable. Evident from this study, many groups struggled to make reasonable assumptions about how many gallons of gas a tank holds, how many miles per gallon an automobile might average, etc. While distance, rate, and time are a common application in middle school mathematics (one in which the students of this study were familiar), the modeling process in this situation requires an even deeper familiarity with cars and driving. What may be perfectly clear ideas and easy applications of mathematics may not always be fruitful modeling contexts; the situations used for modeling need to be more directly related to the lives of students—real-world situations in which the students themselves already have some experiences.

Ultimately, the real work of implementation—and its success or failure—will rest on mathematics teachers, who will need to be able to navigate the two extremes of oversimplification and overcomplication, helping students strike the appropriate balance. For some students, that may mean working on their mathematization (modeling step 2); and for others, that may mean determining which assumptions and variables are most important to keep in the model (modeling step 1). For both groups, however, making revisions to the model (modeling step 5)—a second iteration to improve an oversimplified or overcomplicated model—will need to be emphasized as a critical component of the modeling process. Highlighting the revision process, gathering information about the real world from the model, and comparing it to the actual real-world situation may even help students recognize how their initial models were either overly-simplified or overly-complicated. Awareness of these two tendencies when incorporating modeling in the mathematics classroom should inform teachers as they help students navigate the appropriate balance in the process of mathematical modeling.

Conclusion

The mathematical modeling process requires that students make use of skills that are rarely used in mathematics classrooms outside of the practice of modeling, but which are crucial for applying mathematics to real-world situations. In mathematics education, incorporating the process of modeling poses some potential difficulties. For the implementation of modeling tasks within classrooms to be successful, teachers need to be attuned to students' tendencies as they engage in these tasks. This study contributes to understanding some of those inclinations, in particular to create overly-simplified or overly-complicated models. Different aspects of the modeling cycle—especially making assumptions, mathematization, and revision—may require teachers to address each of these issues in a unique way, guiding students in the development of productive modeling habits. Familiarity and experience with the context is also crucial. Awareness of and attention to these key ideas may help teachers more smoothly transition to incorporating modeling activities in their own classrooms and to helping students strike the appropriate balance in modeling.

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