TABLE OF CONTENTS

JOURNAL OF **MATHEMATICS EDUCATION** AT TEACHERS COLLEGE | FALL/WINTER 2014 | VOLUME 5, ISSUE 2

Using Dynamic Software to Address Common College Calculus Stumbling Blocks

Alice W. Seneres
Rutgers, The State University of New Jersey

John A. Kerrigan
Rutgers, The State University of New Jersey

ABSTRACT There are specific topics in college calculus that can be major stumbling blocks for students. Having taught college calculus for four years to over a thousand students, we observed that even the students who have already taken pre-calculus or calculus during their high school careers had common misunderstandings. Students may remember a technique without retaining the understanding of why it can be applied or what it is accomplishing, essentially only having knowledge of a rote procedure. Educators can address these areas of difficulty by regularly utilizing dynamic technologies such as Geometer's Sketchpad and Desmos in the classroom to fully illustrate calculus concepts. With these tools, teachers can help their students better understand how to reason mathematically in calculus.

KEYWORDS *college calculus, AP calculus, graphing software, teaching calculus theorems, student expectations, dynamic, questioning*

Introduction

Over half of the students who enroll in a college Calculus I course have taken calculus in high school (Bressoud, Carlson, Mesa, & Rasmussen, 2013). Yet despite familiarity with the material, certain concepts in calculus elicit common mathematical errors or lapses in reasoning. During the first semester of teaching college calculus, there was an assumption on the authors' part that students who had taken calculus in high school would easily grasp the material. After four years of teaching calculus we realize that even the students familiar with the concepts still experience common misunderstandings. We will examine several of these common calculus stumbling blocks and discuss how teachers can address these misunderstandings when presenting the material.

Definition of a Derivative

In our experience, many students view the definition of the derivative as a rote procedure, and a complicated one at that. Students often breathe a sigh of relief when the derivative shortcuts are introduced and wonder why they even had to learn the formal definition in the first place. However, not grasping the underlying concept of a derivative as the limit of a difference quotient can lead to a lack of understanding of several fundamental concepts in calculus. Students who do not internalize these concepts may struggle with later material, such as rates of change.

The tendency to view the formal definition procedurally is understandable. Students begin working with function notation as early as middle school when they define, evaluate, and compare functions (National Governors Association Center for Best Practices & Council of Chief State School Officers [NGA Center & CCSSO], 2010). As students begin the journey toward calculus, they first come to understand a function as an input/output machine. They are given a rule and have to use the rule to find an answer to a question. Once students reach algebra 2 and pre-calculus, they begin to see functions as expressions to be manipulated or simplified (NGA Center & CCSSO, 2010); indeed many pre-calculus students manipulate a difference quotient at some point as they learn more advanced factoring and expansion techniques.

In contrast, calculus is dynamic; behavior is not only examined at a specific and static point, but as a certain point is approached, or as the distance between two points approaches zero. This can be challenging, since "students frequently treat variables as symbols to be manipulated rather than as quantities to be related" (White & Mitchlemore, 1996, p. 91). Making this leap in understanding is essential; virtual graphing software can help students see the bigger picture (National Council of Teachers of Mathematics [NCTM], 2014). The Desmos software's slider capabilities are helpful in presenting calculus and mathematics more dynamically to students and in ways that would be difficult with just a chalkboard.

In Figure 1, a parabola and its difference quotient function are graphed on the same screen. The student first creates a line by choosing the desired point of tangency and a second point some distance *(h)* away. By then dragging the second point closer to the point of tangency, and seeing the effect it has on the behavior of the secant line, the student gains an insight about relationship between limits and derivatives. The student also gets a feel for what the parameter *"h"* does by moving the second point closer and closer to the point of tangency.

As students explore with sliders, teachers can encourage mathematical discourse by asking guiding questions such as "What happens as *h* approaches zero? What happens when *h* is exactly zero?" (NCTM, 2014). These questions help students understand the important relationship between secants, limits, and the tangent line. Ideally this conceptual understanding will result in a procedural fluency that the students retain even when the visual representation is no longer available.

Even when a visual representation is available, teachers and students must use graphing software with caution. Graphing software may not clearly indicate asymptotes or holes the way students are accustomed to seeing in their textbooks. To that end, both teachers and students are encouraged to not use graphing software exclusively, but to augment the visual representation with both tables and algebraic analysis of functions. This will give a true picture of the domain.

Utilizing Theorems

Students are first formally introduced to theorems and proofs in their high school geometry class (CCSSO, 2010), and initially may see them as routine exercises.

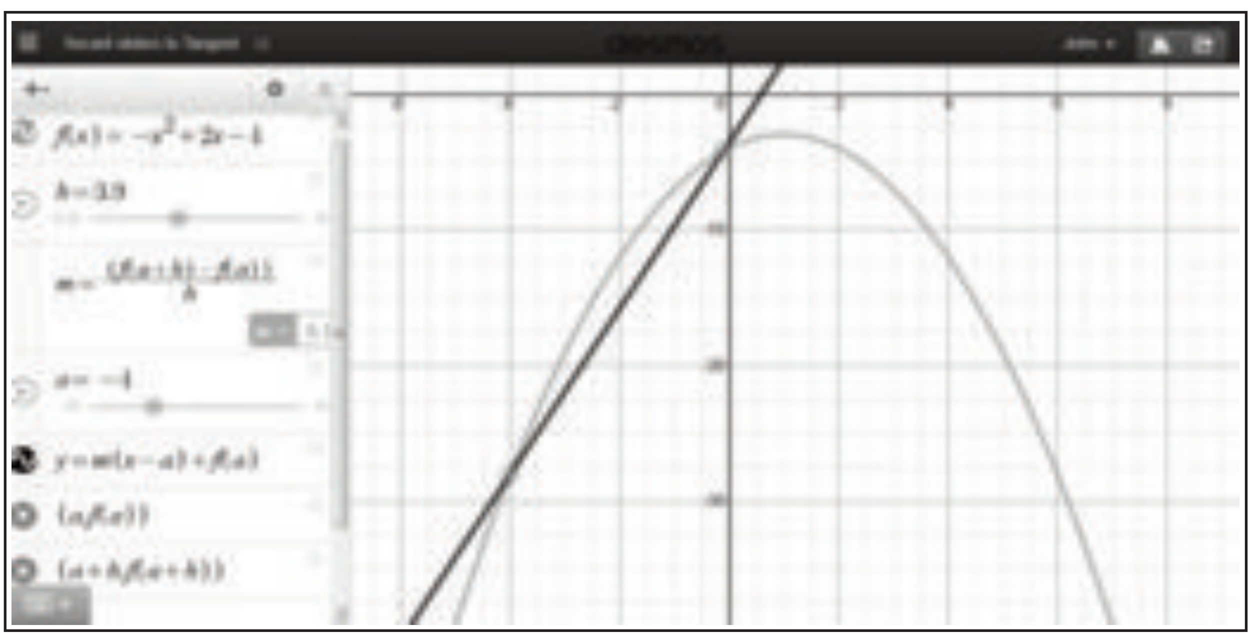

Figure 1. Desmos Software graphing a function and its difference quotient

These types of exercises can sometimes lack the visual, dynamic, and active representations of functions that students need to embrace in a calculus course. In addition, in a calculus course students are required to critically analyze and check the assumptions and given information before applying a theorem or executing a proof. After working through a series of logical steps, students then also have to interpret their results in the context of the problem. This new, added workload makes applying theorems and executing proofs difficult for some first-semester calculus students. Specific examples are discussed along with strategies for improving understanding.

Intermediate Value Theorem

The Intermediate Value Theorem is a beautiful existence theorem that is introduced at the high school level in pre-calculus or calculus. However, we have found that students can find it challenging because questions invoking the theorem require students to understand and prove why a root must exist given certain conditions.

Students have spent significant portions of their mathematical lives finding the exact root(s) of a function using algebraic methods or the graphing calculator. The Common Core State Standards for High School Algebra place a heavy emphasis on solving equations as a process of reasoning (CCSSO, 2010); many students enter college well-equipped to solve various types of equations in their mathematics courses. We have found that many students feel uncomfortable stopping at stating that a root exists for a given function on an interval. The exercise may seem pointless to them without having a specific value at the end of the problem they can point to. Understanding why it is useful to merely identify if there exists a root can help students gain confidence in utilizing this theorem.

The Intermediate Value Theorem also sometimes gets applied blindly by students who do not first check for continuity. It is possible that they simply assume that during the "Intermediate Value Theorem section" of the course, the functions they are asked to work with must be continuous or the Intermediate Value Theorem could not be applied. Yet thinking about the curriculum in this compartmentalized fashion can be problematic, and it is necessary to become accustomed to rigorously checking that all the conditions are being met before utilizing a theorem.

This mindset can have implications far beyond a mathematics classroom and extends to real-world mathematics and STEM projects, whereas a simple oversight can have serious consequences. A project could be assigned for each student to research and present real-world examples where assumptions were made that ultimately did not hold, and what the repercussions were. This emphasizes the importance of checking all assumptions, and provides the students with an opportunity to explore an issue in their area of interest.

As an example, consider the space shuttle Challenger, which exploded after take-off and killed all seven crew members. A review found that no one involved in making the final decision to launch in the cold weather had data on how the O-rings would perform at such low temperatures (Vaughan, 1990). This is a stark example that a teacher could provide to illustrate the importance of checking that all conditions are met.

Some students may not check for continuity because they do not know how to determine if a function is continuous without graphing it. Other students may not realize the need to ensure that the function is continuous, which points to a lack of understanding of the theorem itself. A non-continuous function could meet all of the other requirements showing the existence of a root, when a root in fact does not exist. A student who cannot explain why a function needs to be continuous often benefits from seeing a graph of this case.

For example, a college calculus exam problem involving the Intermediate Value Theorem could state: "Determine if $\frac{1}{x} = x^3 - 1$ has a solution on the interval $[-\frac{1}{2}, \frac{1}{2}]$." Students who do not carefully examine the continuity of the related function $f(x) = \frac{1}{x} - x^3 + 1$ can incorrectly conclude that since $f(-\frac{1}{2}) < 0$ and $f(\frac{1}{2}) > 0$ then $f(x) = \frac{1}{x} - x^3 + 1$ must have a root on the prescribed interval. Graphing software tells a different story (Figure 2).

Our students are usually surprised to realize their conclusion was wrong when they see the presence of a vertical asymptote; the function never crosses the x axis on the given interval $[-\frac{1}{2}, \frac{1}{2}]$.

Problems such as these often pose the greatest challenge for students, especially on examinations. Projecting the graph and carefully questioning students while reviewing the problem can help students understand the danger of blindly applying theorems.

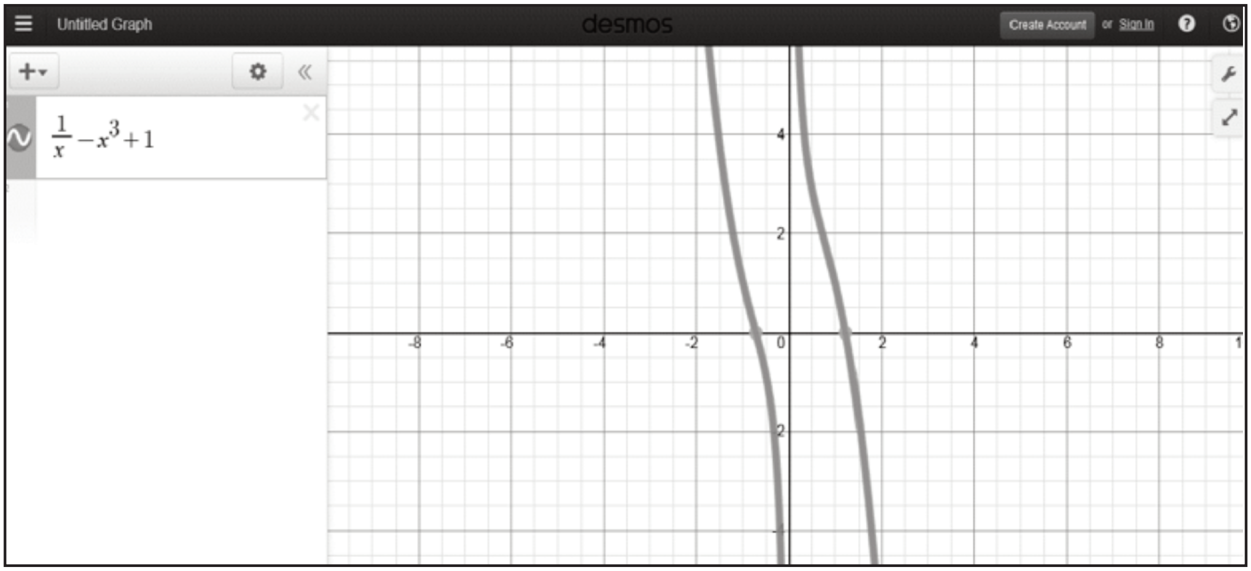

Figure 2. Desmos Software graphing a discontinuous function

Mean Value Theorem

Often the Mean Value Theorem is introduced as the location on the function where the tangent line would be parallel to the secant line segment. Indeed, the Advanced Placement Calculus curriculum specifically stresses that students must understand the geometric consequences of the Mean Value Theorem (College Board, 2012). Dynamic software such as Geometer's Sketchpad allows teachers and students alike to visualize this; students enjoy being able to physically drag the tangent line to a location where it would be parallel to the secant segment.

Once students have this understanding, the discussion can evolve into how the Mean Value Theorem identifies when the average rate of change of a function is equal to its instantaneous rate of change. The difficulty arises when students have to algebraically find the value of c to satisfy the conclusion of the Mean Value Theorem. Students often stop a step short by setting $f'(c)$ equal to the slope of the secant line segment and calculating the slope of the secant line segment. Yet this will only tell the student what the slope of the tangent line needs to be; a student stopping at this step has not determined where on the function this will happen.

Failure to complete the process points to confusion as to what exactly is being determined when the Mean Value Theorem is applied, and what the value of c really represents. Linking the algebraic work back to the graph in Figure 3 is a way to remedy this type of misunderstanding. Having to move the tangent line to find a location where the slope is equal to the slope of the secant line emphasizes that students are finding exact coordinates, specifically the location where the tangent line and secant line are parallel.

In addition, we have seen that many students often forget to check the differentiability and continuity requirements. Again, visualizing certain functions that do not meet these requirements—yet would (falsely) allow a student to solve for a value of c that actually does not produce a tangent line parallel to a secant line segment—is beneficial. Consider the function $f(x) = x^{2/3}$ on the interval [–1, 1] in figure 4.

Without examining the graph and its non-differentiability at $x=0$, students can potentially set up a calculation and arrive at an answer that does not make any sense.

Finally, students tend to also have difficulty distinguishing between the hypotheses and conclusions of the Intermediate Value Theorem and Mean Value Theorem. The Scoring Commentary from the 2013 AP Calculus AB examination revealed that given an applied situation about rates, the most common (incorrect) answer was the Intermediate Value Theorem, instead of the Mean Value Theorem (College Board, 2013b).

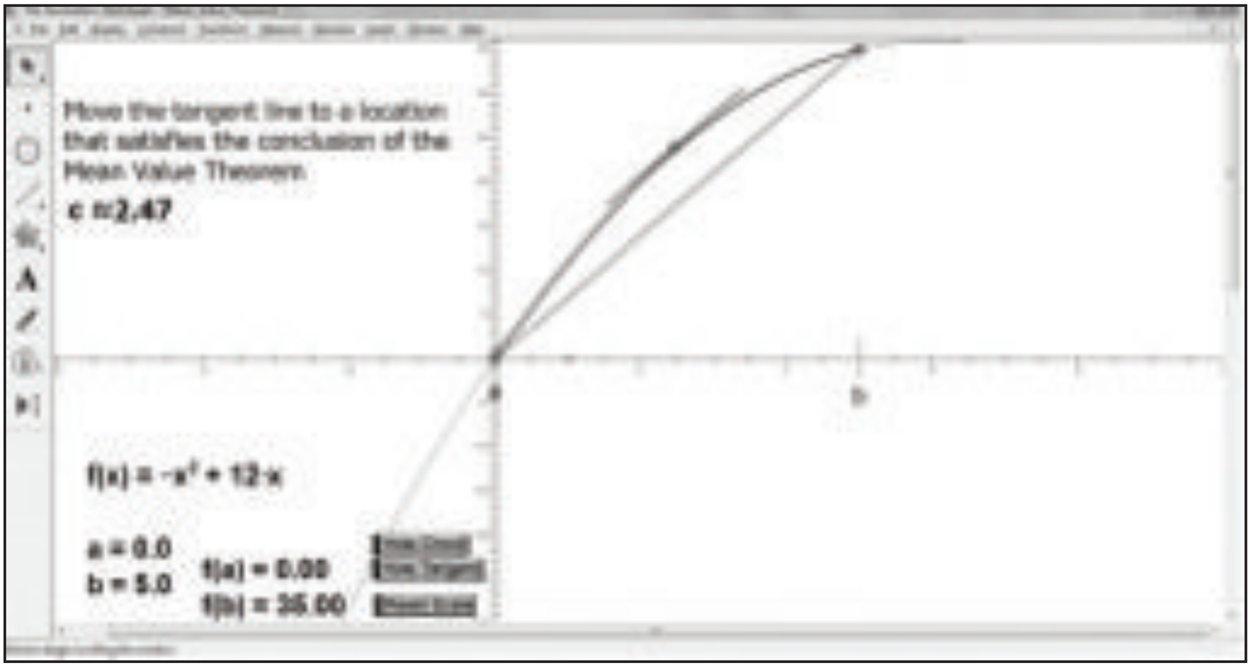

Figure 3. Geometer's Sketchpad Software visualizing a function's tangent and secant lines

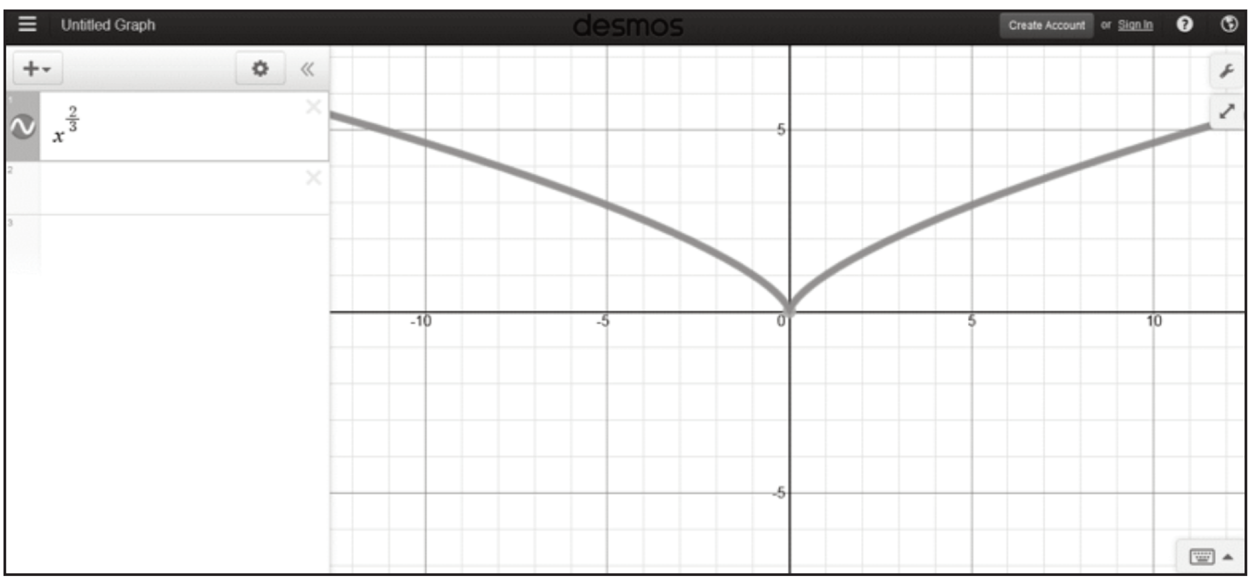

Figure 4. Desmos Graphing Software graphing a function with a cusp

Question 3

t (minutes)	0	1	2	3	4	5	6
C(t) (ounces)	0	5.3	8.8	11.2	12.8	13.8	14.5

Hot water is dripping through a coffeemaker, filling a large cup with coffee. The amount of coffee in the cup at time t, $0 \le t \le 6$, is given by a differentiable function C, where t is measured in minutes. Selected values of C(t), measured in ounces, are given in the table above.

(a) Use the data in the table to approximate $C'(3.5)$. Show the computations that lead to your answer, and indicate units of measure.

(b) Is there a time t, $2 \le t \le 4$, at which $C'(t) = 2$? Justify your answer.

Figure 5. AP Calculus AB Question #3 on 2013 exam (College Board, 2013a)

Given the similar hypotheses of both the Intermediate Value and Mean Value Theorems, many students saw part (b) of the question in figure 5 as an opportunity to apply the Intermediate Value Theorem. Since students had to provide an estimate for $C'(3.5)$ in part (a), it seemed only natural to do the same for part (b) by thinking of the Intermediate Value Theorem in reverse. Once again, a deeper knowledge of the hypotheses of the theorems would help a student realize that differentiability is associated with the Mean Value Theorem, not the Intermediate Value Theorem. For this reason it is helpful when teaching the Mean Value Theorem to take some extra time to compare and contrast it to the Intermediate Value Theorem.

L'Hopital's Rule

Another very powerful theorem in calculus is l'Hopital's Rule; when this topic is initially taught, often virtual graphing software is used to illustrate the need for the rule. At this time, teachers revisit the idea of a "hole" or discontinuity and tie it into the new concept of an indeterminate form.

It is important to note that holes are not always properly represented with graphing software, since these devices plot discrete points. Functions with holes may initially appear to be continuous when in fact they are not. This is also an opportunity to discuss the importance of not over-relying on technology, but using it as a tool to augment other approaches. When a particular device lacks the capability to initially illustrate a hole, the points where a discontinuity may occur can be determined algebraically. The Desmos software, as shown below, then allows the user to "click" on a suspected hole to reveal its coordinates. Consider the problem, $\lim_{x \to 0} \frac{\sin x}{x}$, as shown in figure 6.

Once the need for l'Hopital's Rule has been shown, our students are typically excited to learn that they

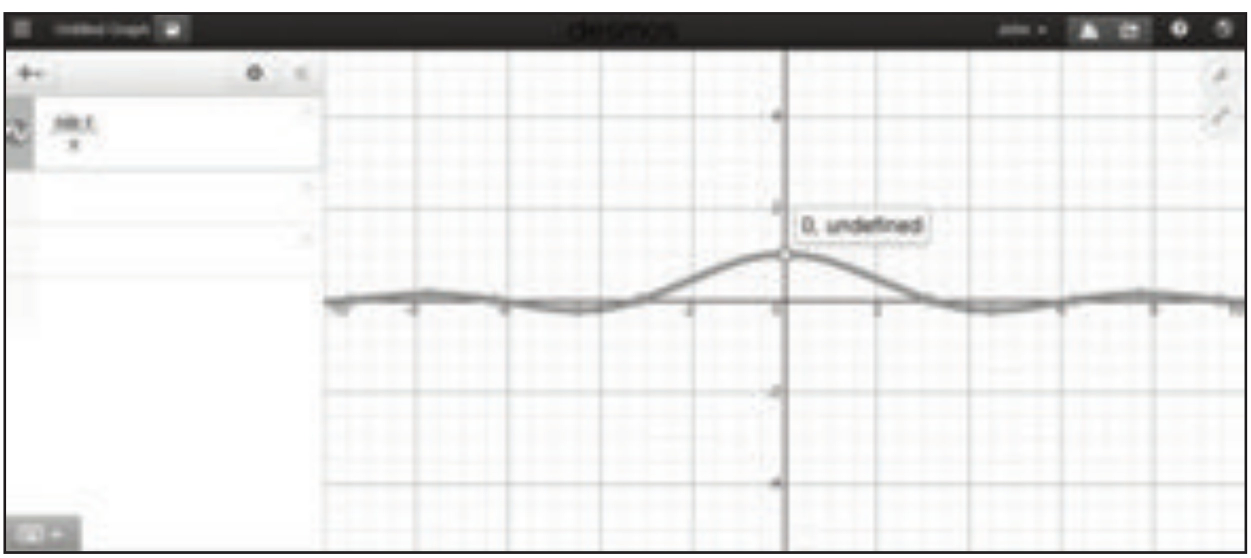

Figure 6. Desmos Software graphing a function with a hole.

can skip a messy geometric proof and simply take the limit of the derivative of the numerator over the derivative of the denominator to determine the value of the limit. The ease of the shortcut sticks with students and unfortunately may cause them to blindly apply l'Hopital's Rule without checking for an indeterminate form.

Forgetting to check for an indeterminate form can have consequences. Take for example $\lim_{x\to 0} \frac{\sin x}{\cos x+1}$. In the beginning of a calculus course, many students learn ways to algebraically manipulate limits like this by using multiplication and trigonometric identities. We have seen that once the semester progresses, students may instead go straight to using l'Hopital's Rule, and obtain a completely incorrect answer since the function did not have an indeterminate form.

It is worth noting that l'Hopital's Rule is not on the list of Advanced Placement (AP) topics and does not appear on the AP test. Because of this, some students will come out of high school calculus having never learned it. If they go into college calculus confident that they have seen all of the material before, they may be surprised to discover this portion of the course will not be a review for them, but will be new—and important—material.

Curve Sketching

While curve sketching is a relatively straightforward practice to grasp, the reality is that even the strongest students can struggle with it, though the building blocks are familiar (Baker, Cooley, & Trigueros, 2000). Students entering calculus have worked with the domain, range, intercepts, and asymptotes of a function many times before. In calculus, the connection is made between the concept of a limit as x approaches infinity and a horizontal asymptote; the limit is identifying the long-term behavior of the function. Yet when students are asked to determine the horizontal asymptote, in our experience they often revert back to prescribed rules about degrees and leading coefficients; we find that they are not as comfortable using a limit argument. This may point to a lack of understanding about what exactly a horizontal asymptote is showing, and why a function may cross a horizontal asymptote. These minor confusions can cause trouble when sketching the function.

Curve sketching also requires students to determine where the function is increasing or decreasing. While mathematically this is not difficult, our students sometimes mistakenly believe that a function's graph cannot only increase or only decrease, but must exhibit both behaviors. Illustrating a monotonic function such as $f(x) = \arctan(x)$, can be illuminating (figure 7).

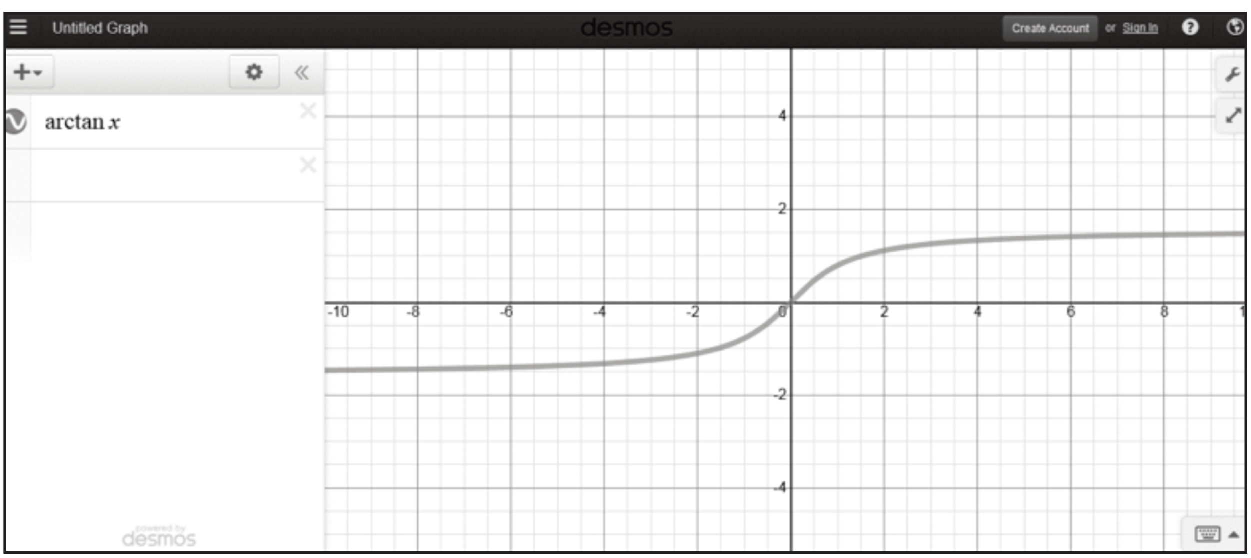

Figure 7. Desmos Software graphing a monotonic function.

When learning how to identify where a function is increasing or decreasing, it is also worth examining the behavior of a function with vertical asymptotes. This provides the opportunity to discuss the need to conduct a sign analysis on every interval. Students may try to conduct a sign analysis on only one interval, then assume the behavior changes on the next interval—if it was increasing, it must then switch to decreasing. Of course this assumption cannot be made; a sign analysis must be done for each interval to accurately determine the behavior of the graph, as the function $f(x) = \frac{x}{x^2-1}$ illustrates (figure 8).

The function in figure 8 also addresses another common error we have observed; students can carry out correct supporting work, yet doubt the results when they determine the derivative has the same sign on each interval. If they do not realize this behavior of decreasing on every interval, or increasing on every interval, is possible, they may think their own work is incorrect. This same issue can also arise when determining where a function is concave up and where it is concave down. By becoming familiar with functions that exhibit these behaviors, students can have the confidence to accurately sketch the graph based on their work.

Conclusion

Teaching calculus is very rewarding, as it is an opportunity for educators to excite students about mathematics and address common stumbling blocks that even the strongest students may experience. Technology can be a useful tool to demonstrate the more dynamic aspects of calculus that would be difficult to illustrate on a chalkboard. By employing a visual approach that includes dynamic software, students can make connections between the behavior of the function they are examining and the supporting algebraic work they have been carrying out. Constantly linking concepts with graphs as well as procedures, tables, and pointed questions can provide students with the big picture of calculus, as well as an appreciation for all of the subtleties that go along with it. This type of background knowledge is essential for students as they continue taking courses that utilize calculus concepts.

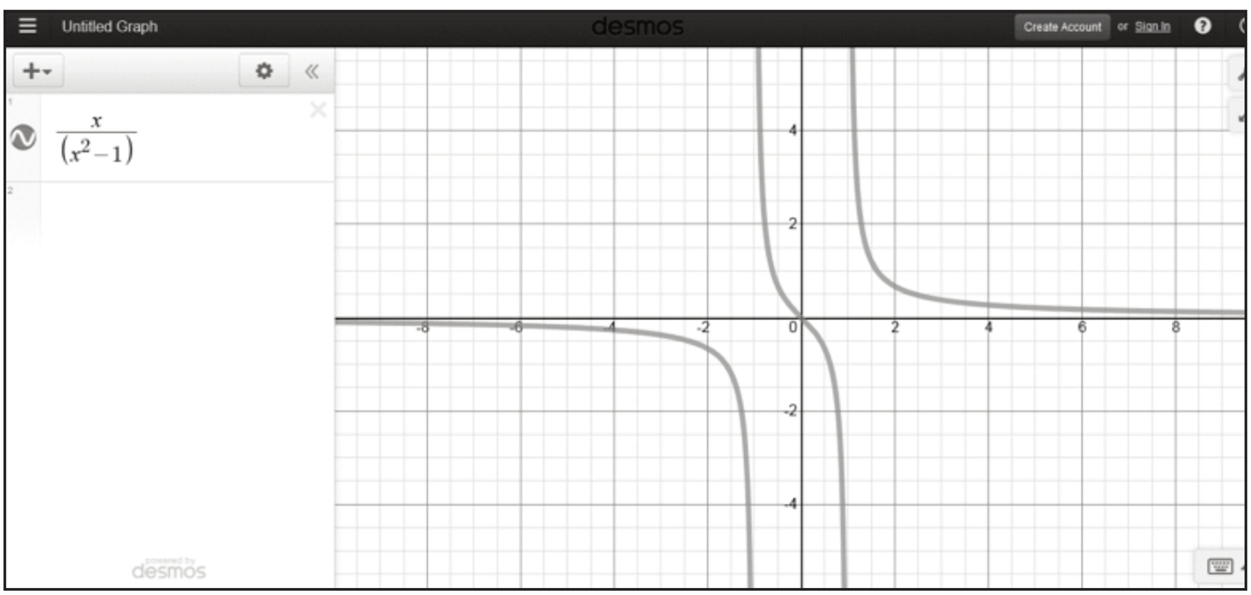

Figure 8. Desmos Software graphing a function decreasing on every continuous interval.

References

Baker, B., Cooley, L., & Trigueros, M., (2000). A calculus graphing schema. *Journal for Research in Mathematics Education, 31*(5), 557 – 578.

Bressoud, D.M., Carlson, M.P., Mesa, V., & Rasmussen, C. (2013). The calculus student: Insights from the mathematical association of america national study. *International Journal of Mathematical Education in Science and Technology, 44*(5), 685 – 698.

College Board. (2012). AP calculus course description. Retrieved from http://media.collegeboard.com/digitalServices/pdf/ap/ap-calculus-course-description.pdf

College Board. (2013a). 2013 AP calculus free response questions. Retrieved from http://media.college board.com/digitalServices/pdf/ap/apcentral/ap13_frq_calculus_ab.pdf

College Board. (2013b). AP Calculus AB 2013 Scoring Guidelines Question 3. Retrieved from http://media.collegeboard.com/digitalServices/pdf/ap/apcentral/ap13_calculus_ab_q3.pdf

Jackiw, N. (2001). The Geometer's Sketchpad [computer software]. Emeryville, CA: KP Technologies.

Luberoff, E. (2011). Desmos [virtual graphing calculator]. San Fransicso, CA: Desmos Inc.

National Council of Teachers of Mathematics. (2014). *Principles to actions: ensuring mathematical success for all.* Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common core state standards for mathematics.* Retrieved from http://www.corestandards.org

Szydlik, J.E., (2000). Mathematical beliefs and conceptual understanding of the limit as a function. *Journal for Research in Mathematics Education, 31*(3), 258 – 276.

Vaughan, D. (1990). Autonomy, interdependence, and social control: NASA and the space shuttle challenger. *Administrative Science Quarterly, 35*(2), 225 – 257.

White, P., & Mitchelmore, M., (1996). Conceptual knowledge in introductory calculus. *Journal for Research in Mathematics Education, 27*(1), 79 – 95.