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MATHEMATICAL
MODELING**

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Editors:

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Mathematical Modeling, Sense Making, and the Common Core State Standards

Alan H. Schoenfeld
University of California

On October 14, 2013 the Mathematics Education Department at Teachers College hosted a full-day conference focused on the Common Core Standards Mathematical Modeling requirements to be implemented in September 2014 and in honor of Professor Henry Pollak's 25 years of service to the school. This article is adapted from my talk at this conference covering Mathematical Modeling in the Common Core, what it means for educators as we move toward this future reality, and what we can do to succeed in this brave new world.

Keywords: mathematical modeling, Common Core State Standards, sense making, formative assessment

Note: This is a reprint of the lead article from the Fall 2013 issue of the *Journal of Mathematics Education at Teachers College*. The slides from this talk can be found online at <http://journals.tc-library.org/>.

Thank you so much for the introduction, Henry [Pollak]. It's an honor and a pleasure to be here, to help celebrate Henry's 25 years at Teachers College and, more importantly, to celebrate the broad sweep of his contributions to mathematics education—in his “second career.”

In listening to this morning's proceedings I thought of two additional introductions to this talk. Now I have three, and they're brief, so I might as well begin this presentation with all of them. First, you may have noticed that Bruce Vogeli introduced the break before my talk by saying “It's the tradition in my country to have people spread manure;” I was sure he was going to say “And next we're going to hear from Alan.” I'm grateful for his restraint.

Second, Werner Blum and I were talking about how beautifully organized Rita Borromeo Ferri's talk was. That's a counterpoint to my kind of talk. Henry has described my kind of talk: he once described a presentation as “a series of digressions tied together by non-sequiturs.” I'll do my best not to live up to that description, though I'm not off to a good start.

Third, you may have noticed that Bruce started this morning's session with a list of eminent mathematical modelers, including many people in this room. My name was not among them. That's fair, if one thinks in traditional mathematical terms—people tend to think of models as pertaining to “real world” situations, for example modeling traffic flow, or heat diffusion, or predator-prey relationships. I don't do that. So, earlier this morning I trolled the web looking for a slide that addressed the question: “What's a guy like you doing in a place like this?” Here's what I found. (Projects a slide of a wolf in sheep's clothing.)

I do want to note that in my serious work—see my book *How We Think* (Schoenfeld, 2011)—I do build precise models of people's decision making. That work is decidedly mathematical, and reflects my mathematical roots. As Henry has noted, there are no theorems in mathematics education—but, one can still try to build models of all sorts of phenomena, including models of what goes on inside people's heads.

Now let me tell you what this talk is about. In part 1 I want to give some background. The Common Core (CCSSM, 2010) is about to become a reality, and as I see it, the Common Core gives us license to do the kinds of things we've wanted to do ever since the NCTM *Standards* appeared in 1989. But make no mistake, it's hard. So parts 2 and 3 of my talk are going to be about things that will help: tools for the classroom and tools to help us reflect on our teaching. And, if I can reach back to my New York heritage and talk fast enough, we'll actually have some time to chat at the end.

So where are we? As far as I'm concerned, the purpose of mathematics education has always been to help students to do mathematical sense making. That's really what the 1989 NCTM *Standards*, which were grounded in the research we did from the mid-1970s, were all about. If you think about it, the major difference between the '89

Standards and the curriculum desiderata that preceded them was that the *Standards* expanded beyond descriptions of content (arithmetic, number, measurement, geometry, data, etc.). Those were all in the *Standards* but they were preceded in every grade band by the following: problem solving, reasoning, connections, and communication. Those are the processes of *doing* mathematics. That's where the action is. But to set the context I want to start where the action isn't.

Here is a “modeling” problem, which has been used in research all over Europe. The problem is: “Alan has a little corner where he wants to build a bookshelf. It's just going to be two feet wide. He has two five-foot long boards. How many two-foot sections can he get from those boards?”

The answer, of course, is four. Each board produces two two-foot shelves, with a one-foot board left over. Now the key question: “What did seventy percent of the students worldwide asked this question say the number of two-foot boards is?”

Several audience members: “Five?”

Yes, five. Why? Because five and five are ten, and ten divided by two is five—end of story. We all know that in reality, word problems in math classrooms are used as cover stories for arithmetic. From the student's perspective (based on their classroom experience!) they have nothing to do with the real world or modeling.

What follows by way of elaboration of this theme are some oldies but goodies. Kurt Reusser asked ninety-seven first and second graders: “There are twenty-six sheep and ten goats on a ship. How old is the captain?” You laugh. But the fact is that seventy-six of the ninety-seven students solved the problem. Their thinking went something like this: “Well, let's see, twenty-six and ten, that's thirty-six. Yeah, the captain could be thirty-six. Twenty-six minus ten, that's sixteen. Naw, that's too young. Twenty-six times ten, naw, he'd be dead. I don't know how to do division, so he must be thirty-six.”

Better yet, Heinrich Radatz tells a bunch of kids this story: “Mr. Lorenz and three colleagues started at Bielefeld at 9 AM and drove the 360 kilometers to Frankfurt with a rest stop of 30 minutes.” Radatz doesn't ask a question, you notice. He just tells a story. He tells it to a bunch of kindergartners and they all go “Thank you for the story.” They're probably thinking, “It's not a very exciting story, but he's a grown-up and if he wants to tell it to us, that's fine.” Radatz tells this story to a bunch of first graders. A few of them combine some of the numbers and give him an answer—even though there was no question! He does that for second graders, third graders, fourth graders, all the way to sixth grade. And every year more kids than the year before combine the numbers and give an answer. Why? Because what you learn about math in school is that your job is to combine the given numbers and produce an answer. Alas, math in school is often not about sense making.

In contrast, here's a sense making example that comes from Deborah Ball's third grade class. Deborah had a bunch of kids who were playing with numbers—adding, subtracting, doing various things. They noticed that every time they added two odd numbers, they got an even number. One of the kids said, “Is that always going to happen?” She then stopped and said, “But the odd numbers go on forever. We can never test them all, so we can never know.” That's pretty damned good for a third grader.

Another kid says, “Well, I was thinking about seven plus nine, and actually I knew that the sum was going to be even before I added them up. If you look at seven, it's a bunch of pairs with one left over, and if you look at the nine, it's a bunch of pairs with one left over (Figure 1). So when you put them together, the pairs are going to stay the same—but the two leftovers become a pair, so everything together is in pairs, so it's going to be even” (Figure 2). The girl stopped at that point and she said, “But wait! It doesn't have to be seven and nine. No matter what that first odd number is, it's going to be a bunch of pairs with one left over. The second odd number is also going to be a bunch of pairs with one left over. And when you put them together, you have all the pairs you started with plus the pair you made, so it's going to be even.”

Now that's a third grader. I guarantee you that every mathematician I know would say that the student produced a completely rigorous mathematical proof. That's what I call mathematical sense making, and that's what I'd like to see in our classrooms. The real challenge we face is to support sense making in our classrooms. In the context of this talk, that means discussing how sense making is related to the Common Core and everything that comes with it—including modeling, of course.

But first, in the spirit of one of my introductions, a digression. That is: What's the relationship between problem solving and modeling? Earlier we heard Dick Lesh invoked—Dick has done a lot of work on modeling. Dick's point of view is that classic math education has a distorted view of problem solving and modeling, because it views applied problem solving (a.k.a., modeling) as a subset of traditional problem solving, while it should be the other

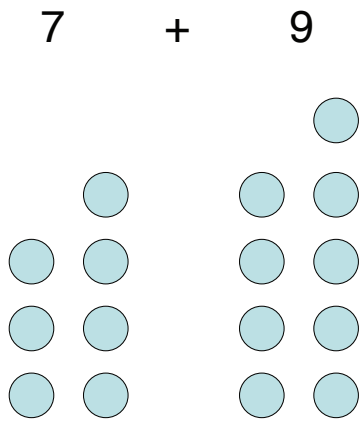


Figure 1.

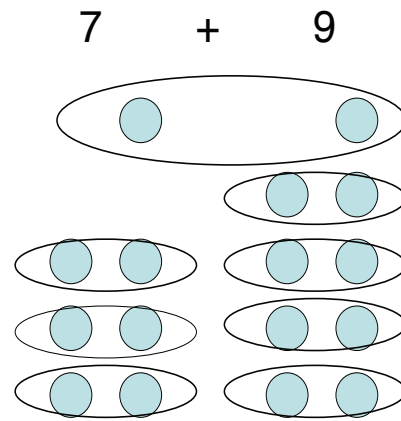


Figure 2.

way around. He believes one should think of applied problem solving as a modeling activity, and then traditional problem solving as a subset of that.

I'd rather think about it this way. If you think about traditional problem solving in the right way, it's about taking some mathematical phenomenon, figuring out what makes it tick, delving into it, perceiving order and structure—that is, engaging in sense making—and doing what you need to do to achieve your goal. If you think about applied problem solving, it's taking some real-world phenomenon, figuring out what makes it tick, delving into its structure, and then doing whatever mathematics you need to do to make sense of it (Figure 3). So why not call them both mathematical sense making, and say that's our goal? What we're really trying to do is figure out how things work. Sometimes the structure is purely mathematical, sometimes the structure is motivated by the real world. But it's the same game.

That said, let me turn to the announced topic, the Common Core State Standards. They have two main foci: content and practices. The key words regarding content are *focus and coherence*. If you were to hear Phil Daro (one of the Common Core authors) talk, the children of America wandered through the desert of fractions for forty years. The authors wanted to focus the curriculum, so, for example, they emphasized fractions as a pathway to proportional relationships, which are the heart of linear functions—a straightforward progression to powerful mathematics.

And yes, the content is good stuff. But I want to focus on the *practices* (which are the natural descendants of the “processes” in the NCTM *Standards*). The practices in the Common Core standards—make sense of problems and persevere in solving them, reason abstractly and quantitatively, construct viable arguments, model with mathematics, use appropriate tools strategically, and so on—are the things you do when you're *doing* mathematics. That's where the action is.

So now a question: “What do the words in the Common Core standards actually mean?” I know this seems odd, but bear with me. Do you know the expression WYTIWYG? It's due to Hugh Burkhardt, whom you'll hear from later. WYTIWYG stands for “What You Test Is What You Get.” To put things bluntly: If a high stakes test is lousy the standards won't mean a thing, because teachers will spend class time drilling the kids on what's on the test. Thanks to No Child Left Behind (2001), we've had the tyranny of testing for better than a decade.

I think it's best to think about things this way:

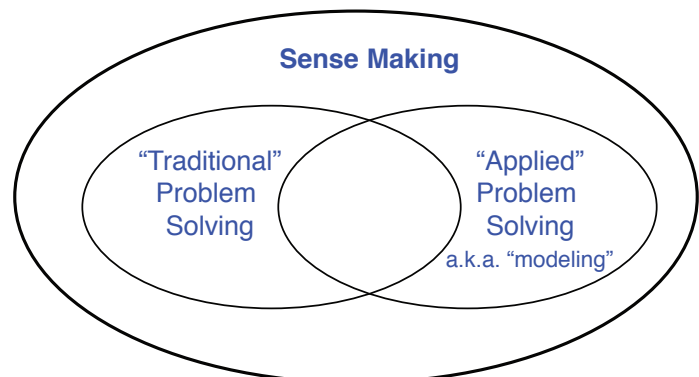


Figure 3.

Tests can be a positive or negative force. I live in California, where they've been a strongly negative force for the past decade. I'm about to show you released items from the California state tests. They're high stakes: if you want to go to the next grade; if you want to get a diploma; if you don't want your school to be ripped apart; then you have to do okay on these tests.

First problem: "What's the y -intercept of the graph $4x + 2y = 12$ " (Figure 4)? This is considered a "difficult" two-step problem because either you've got to remember to plug in $x = 0$ and then solve the (oh so) difficult equation $2y = 12$, or put the equation in standard form and read off the intercept.

Second problem: "Which of four candidate graphs best represents the graph of $y = 2x - 2$ " (Figure 5)? The equation is already in standard form. It has a positive slope and its y -intercept is below the x -axis, so all you have to do is see which of the four graphs has those properties—thirty seconds and you're done. The third problem (Figure 6) is equally trivial.

Does this matter? *Yes.* Visit California classrooms in February and March, in most of them there's nothing going on except practice on this kind of item. There's no real math being taught—certainly no sense making—because kids, teachers, and schools are being held accountable to this kind of testing.

23 What is the y -intercept of the graph of $4x + 2y = 12$?

- A -4
- B -2
- C 6
- D 12

CSA00299

Figure 4.

25 Which *best* represents the graph of $y = 2x - 2$?

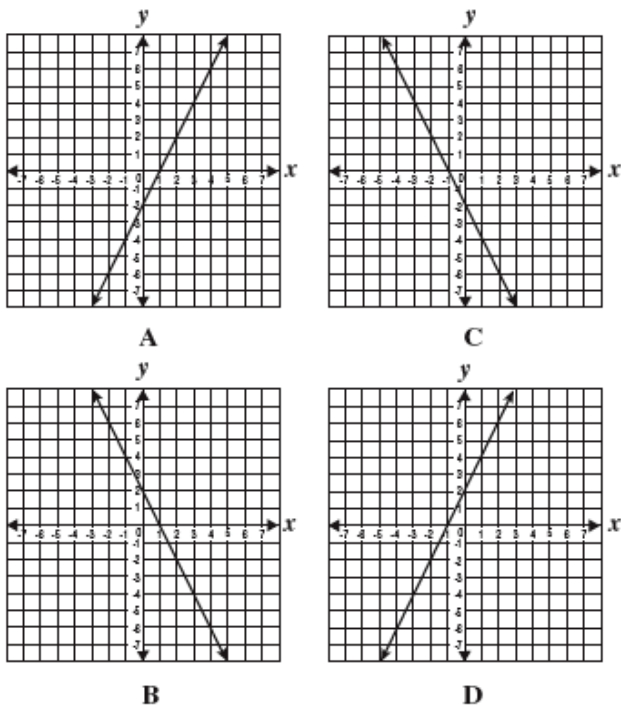
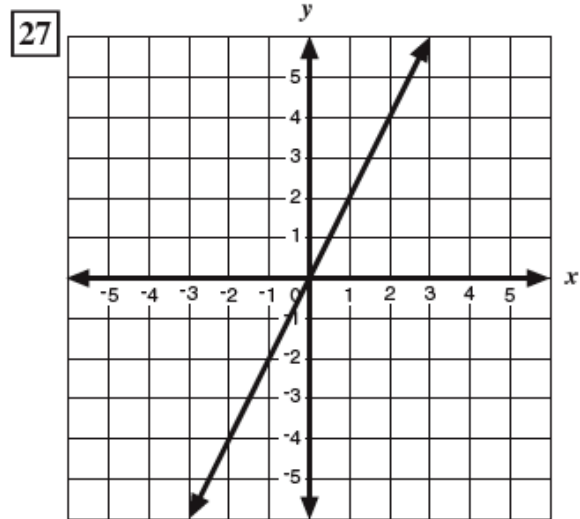


Figure 5.



Which equation *best* represents the graph above?

- A $y =$
- B $y = 2x$
- C $y = x + 2$
- D $y = 2x + 2$

Figure 6.

The point is that the previous tests, which drove instruction (in California at least), were skills-oriented. The Common Core State Standards will demand more. The rest of the talk is about that.

There are two Common Core mathematics assessments, produced by PARCC (Partnership for Assessment of Readiness for College and Careers) and SBAC (Smarter Balanced Assessment Consortium). Here’s the important thing, which plays out differently for the two consortia.

Both consortia pay homage to concepts and procedures, problem solving, reasoning, and modeling with mathematics, but they will do it differently. PARCC says that all four will be measured, but in the end, the PARCC assessment assigns each student a single score. When you look at the score the student gets, what does the number tell you? [Pause] Right. Nothing. It may give you a percentile, but that’s it. It doesn’t give teacher or student any useful information. The thing that we managed to achieve with SBAC is that SBAC is going to report, for every student, separate sub-scores for each of the following: concepts and procedures, problem solving, reasoning, and modeling.

Now consider the school that’s been drilling their students on skills for the past decade—they’ve gotten good at that. They’re going to get a set of scores back that says, “Your kids did just fine on skills. They got a zero on problem solving, a zero on reasoning, and a zero on modeling.” That’s a powerful message saying that you have to do something different, something more consistent with the Common Core Standards. For that reason, I’m hopeful that the tests will make a difference.

That’s still a hope that will happen; we’ll have to see how it plays out. Both Smarter Balanced and PARCC are shell organizations, in that they establish specs and procedures, but then they issue contracts to large testing corporations to create and administer the tests. You won’t believe what a roomful of psychometricians dreams up, thinking that they’re testing problem solving, reasoning, and modeling with mathematics! We have a math board that’s trying to chip away at that and make sure there’s some mathematical integrity to the tests. So we hope that a non-trivial part of the tests will be devoted to the kind of stuff we haven’t seen tested in California for some years. This is going to be traumatic—but that isn’t news to New Yorkers, who’ve recently changed to new, standards-based tests. New York has been the home of recent testing trauma. When people said, “We’ve gotta start testing what’s really in the standards,” average scores went from fifty percent to thirty percent.

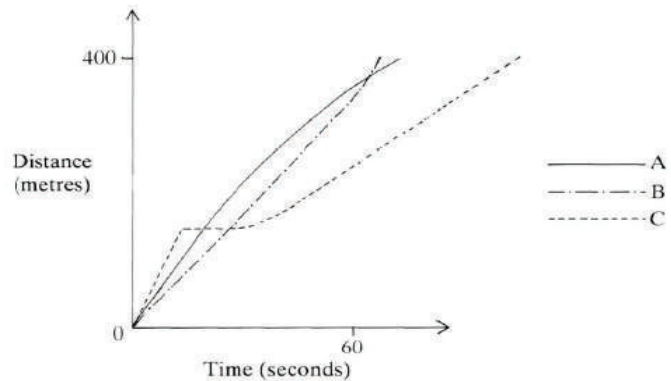
That’s either cause for despair or cause to roll your sleeves up and get to work. My assumption is that for the folks in this room, it’s the latter. So let me give you an example of a test item that I like, that moves in a different direction. This item was written by Malcolm Swan years and years ago. It’s called Hurdles Race, and it says: “Here’s a rough sketch of three runners A, B, and C in a four hundred meter high hurdles race. You’re the race commentator. Write a commentary explaining what happened.” (Figure 7)

Right off, a lot of kids go, “Oh, look, runner C’s graph is way to the right. He must have won, right?”

Oh wait, that’s time on the horizontal axis—so it took C the longest. Now, C’s graph looks weird. What happened during this horizontal segment of his graph? [Several audience members voice different comments]

He swam across a lake? The distance from the start isn’t changing, so he’s not going anywhere. Did someone say he fell? Oh, yeah, it’s a hurdles race—so he must have tripped on the hurdle. Cool.

Now, what happened up there at that point where the other two curves crossed? [Several audience members voice different comments]



The rough sketch graph shown above describes what happens when 3 athletes A, B and C enter a 400 metres hurdles race.

Imagine that you are the race commentator. Describe what is happening as carefully as you can. You do not need to measure anything accurately.

Figure 7.

That's right, A and B are at the same distance from the start, at the same time. That is, they're in a dead heat. C is out of the race. A had been in the lead after C had fallen on the hurdle, but now he's slowing down; B catches up at the point of intersection, and then, with a burst of energy, wins the race.

Now you've got a story you can tell, in answer to the request to write a race commentary. Let's consider what the task demands. Think of the content: interpreting real graphs in a real world context, realizing to the left is faster, interpreting a horizontal graph segment, understanding what the point of intersection means, and so on. That's all doing mathematics. Maybe not in the standard drill-and-kill way, but that's fundamental mathematical content. In addition, writing down the commentary calls for lots of the practices—like communicating with mathematics, explaining yourself clearly, and so on.

The Hurdles Race task is at the middle/high school level. Here's an elementary task. I start with the trivial part, which has been the standard “do the math” task: “There's a sale, and prices are twenty-five percent off. Julie sees a jacket that had been priced at thirty-two bucks. How much is it now?” According to some test developers, this is another two-step problem: because either you have to compute $\frac{1}{4}$ of 32 and subtract it from 32, or realize that 25% off means that the item is at 75% of cost, and then take $\frac{3}{4}$ of 32.

Here's our version of what the task should be. What you've just seen is part A of the task. Part B says: “Some things didn't sell the first week, and the store wanted to get rid of its inventory, so the next week they took another twenty-five percent off the sale prices from the first week. For anything that didn't sell that week, they said, ‘Okay, we'll take off another twenty-five percent from the week 2 price; and in the fourth week we'll take off twenty-five percent from the week 3 price.’”

Audience member: “Then it's for free.”

You've anticipated me. The question is, “There's this moron named Alan who says, ‘Twenty-five percent off for four weeks—it's all free! I'm renting a truck and I'm gonna clean up.’ Is Alan right? If he is, explain why. If he's not, explain why not.” Now *that* problem asks for some mathematical thinking and explaining.

Audience member: “A nice twist would be if you add twenty-five percent, will it be the same price? A lot of people assume that, but the base of reference has changed.”

That's absolutely right. In essence that's the very first problem that appears in John Mason, Leone Burton, and Kaye Stacey's *Thinking Mathematically* (2010). It's a lovely problem. We actually have a formative assessment lesson that deals with that. I'll get to the formative assessment lessons later. And there are loads of other mathematically rich problems. These come from some of the tests that the Shell Center has built. I've been a partner in crime with Hugh Burkhardt and the folks at the Shell Center now for over thirty years.

Actually, given the context, it's time for another digression; I should explain how Hugh and I came to be partners. If you go back to the late 1970s, problem solving was pretty much non-existent. I wrote Jeremy Kilpatrick and said, “Where's problem solving on the 1980 ICME draft program?” He said, “There,” pointing to the only session, out of hundreds of sessions on the program that was devoted to problem solving. And that session was buried in the middle of the program. I said that was outrageous. By the time 1984 came around, problem solving was a big deal. The international organizing committee decided to make problem solving one of the congress themes, which meant they needed two chief organizers. I've been told that Henry Pollak said, “Why don't we have Hugh Burkhardt and Alan Schoenfeld do it? They deserve each other.”

Anyway, we can offer lots more interesting tasks. For example: “Figure out a Ponzi scheme (Figure 8)—why shouldn't you go along with it?” or, “Minimize the amount of material you use for a cylindrical drink can, and tell me whether the resulting can is actually something you can grasp in your hand (Figure 9).” If you want to see more such examples, go to the Smarter Balanced specs. Hugh and I were the lead authors for the specs and there we put lots of good, real-world problems in them. Now the story is: We wrote a forty-page document that said, “Here are descriptions of the kinds of things you want to test, and here are some examples.” Then the psychometricians got ahold of the document, and now it's bloated with test-makers' vocabulary. Don't bother to read all the testing stuff, just look for the sample problems. And look for the Mathematics Assessment Project on the web (<http://map.mathshell.org/materials/index.php>). That's the project that we've engaged in most recently. Google “Mathematics Assessment Project” and you will find we're the number one hit. I'm about to show you a couple of our sample formative assessment lessons. The lessons are designed to address the question, “How do we begin to prepare kids to do well on assessments that ask kids to think mathematically—to know the mathematics, to do problem solving

“Ponzi” Pyramid Schemes



Max has received this email. It describes a scheme for making money.

From: A Crook
 Date: Thursday 15th January 2009
 To: B Careful
 Subject: Get rich quick!

Dear friend,

Do you want to get rich quick? Just follow the instructions carefully below and you may never need to work again:

1. At the bottom of this email there are 8 names and addresses. Send \$5 to the name at the top of this list.
2. Delete that name and add your own name and address at the bottom of the list.
3. Send this email to 5 new friends.

Figure 8.

Bestsize Cans



The Fresha Drink Company is marketing a new soft drink.

The drink will be sold in a can that holds 200 cm^3 .

In order to keep costs low, the company wants to use the smallest amount of aluminum.

Find the radius and height of a cylindrical can which holds 200 cm^3 and uses the smallest amount of aluminum.

Explain your reasons and show all your calculations

Figure 9.

and modeling, and to explain their reasoning?” Last month, there were a hundred eighty-seven thousand downloads of those lessons.

Before proceeding I have to say something about the notion of formative assessment. In essence, summative assessment is testing at or near the end of instruction, when it’s too late to matter to the student and to the teacher, except for assigning scores. It tells you what the student can do after the course is over. The purpose of formative assessment is to find out what the student knows while you can still do something about it.

That’s the idea, and there are some important things to understand about it. The first is that formative assessment doesn’t simply mean giving more tests, more frequently. Testing frequency isn’t the point. The point is that the information you gather from the test has to be useful in helping the teacher figure out what to do next. The second, which is very interesting, comes from a study by Paul Black and Dylan Wiliam called “Inside the Black Box” (1998). You know what happens if you give students their papers back with numbers on them only: the kids crumple them up without reading them, and stuff them in their backpacks. Not surprisingly, they don’t get any better at math. If you give students tests and you write comments on the test papers without scores, the kids actually read the comments and their performance goes up on retests. Here’s the interesting part. If you write loads of comments and also put scores on the papers, the impact is the same as if you’d only written numbers: the papers get crumpled up and put in the backpack. The name of the game is providing students information about the quality of their work and working with them to improve it.

That’s hard, so I want to tell you about a tool we’ve been working on. It’s called the Formative Assessment Lesson, or FAL. A FAL starts with a diagnostic situation, and then gives you something productive to do when the diagnostic situation reveals what your students actually do or don’t understand. Here’s an example of a FAL.

Anyone who’s taught functions and graphs know that lots of students will confuse a picture of the situation described in a story with the distance-time graph of the phenomenon described in the story. Here’s one way one of our Formative Assessment Lessons was designed to address that.

We start out with a problem to help the teacher find out what the students actually understand about the content. In this FAL it’s the following: “Tom walks along a straight road from his home to a bus stop. It’s a hundred sixty meters. The graph shows his journey one particular day. Say what might have happened.” (Figure 10). If you give that task to a random set of eighth and ninth graders, a non-trivial percentage of kids are going to say, “Tom walked up a hill, down a hill, walked up a steeper hill, and waited at the bus stop which luckily was flat.” You laugh, but

it's true. And if you're a new teacher or you're a teacher who hasn't been asking students for that kind of explanation, you're really traumatized and you don't know what to do when that stuff hits you in the face. So the question is: "What can you do?"

A preface to what's to come: I'll simply state here (and discuss this evening) that really effective teaching isn't just about conveying information (a.k.a., "telling"). It's about putting students in a position where they grapple with the content to make sense of it. So saying, "No, that's not a hill, that's a graph," isn't going to help very much. What you have to do is be able to ask questions that will get the student to figure that out.

Now, as anyone will tell you, if you've taught a lesson twenty times and listened to your students as they've grappled with the content, you've most likely heard everything, or nearly everything, that they're going to say—both things that are correct and incorrect. If you collect those understandings and misunderstandings, then you can prepare responses in advance, so that when a kid says something that indicates a particular misunderstanding, you can orient him or her in the right direction (without simply saying "No, that's wrong. Do it this way").

That's what our lesson designers did. They said, "Here are some common things that kids get wrong and here are some questions you can ask." One of my favorites is that kids think that any time there's a horizontal line in a distance-time graph, it means the object in the graph is not moving. A nice question to problematize that misunderstanding is: "What would the distance-time graph of your distance from your house look like if you were walking in a circle, with your house at the center of the circle?" We have lots of suggestions like that (Table 1), which anticipate student responses and prepare the teacher for them.

The lesson itself starts with this task (Figure 11). If we had more time I would take you through the three possible stories that might correspond to this graph. Option B, with Tom riding his bike east up a hill, and then the hill leveling off is one clear distractor, corresponding to the picture-graph confusion I mentioned before.

In the classroom, the teacher has the students think about the three stories and then vote: "Which one of the options corresponds to the graph?" You'll get kids voting for A, B, and C and then explaining why they voted for A, B, and C. In whole class mode, you can discuss the various parts of the graph—for example looking at the

Every morning Tom walks along a straight road from his home to a bus stop, a distance of 160 meters. The graph shows his journey on one particular day.

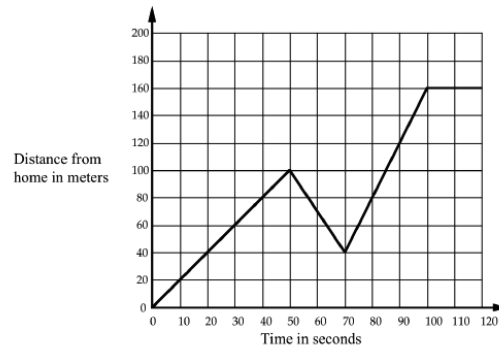


Figure 10.

Table 1.

Common Issues	Suggested questions and prompts
<p>Graph interpreted as a picture E.g. The student assumes that as the graph goes up and down, that Tom's path is going up and down. E.g. The student assumes that a straight line on a graph means that the motion is along a straight path. E.g. The student thinks the negative gradient means Tom has taken a detour.</p>	<ul style="list-style-type: none"> • <i>If a person walked in a circle around their home, what would the graph look like?</i> • <i>If a person walked at a steady speed up and down a hill, directly away from home, what would the graph look like?</i> • <i>In each section of his journey, is Tom's speed steady or is it changing? How do you know?</i> • <i>How can you work out Tom's speed in each section of the journey?</i>
<p>Graph interpreted as speed v time The student has interpreted a positive gradient as speeding up and a negative gradient as slowing down.</p>	<ul style="list-style-type: none"> • <i>If a person walked for a mile at a steady speed, away from home, then turned round and walked back home at the same steady speed, what would the graph look like?</i> • <i>How does the distance change during the second section of Tom's journey? What does this mean?</i> • <i>How does the distance change during the last section of Tom's journey? What does this mean?</i> • <i>How can you tell if Tom is travelling away from or towards home?</i>

first segment of the graph and asking if its right endpoint would be higher or lower if Tom was going faster or slower. You can work through the annotations with the class, annotating the graph.

So far this is pretty much a standard lesson. Here's the cute part. You then break the class into groups and you say, "I'm going to give you ten stories and ten graphs (Figure 12). Your job is to match them and make a poster." A lovely thing about that is you can wander around the class and see what the students have paired together. When you see a student who has one story and three graphs, you know there's one kind of issue. When you see one story and one graph, you can do an immediate diagnosis without saying a word, seeing if it's right and figuring out what's causing trouble if it's not.

So then what? Part of the way through the lesson, when the students have made some progress, you can offer help. You can tell the students, "I know it's hard to see the relationship between story and graph, but you can create something that helps. The graphs don't have units. But, you can assign units, and then make a table that shows how the distance and time change. When you do, it's often easier to see which story the graph represents. Now I'm going to give you ten tables, eight of them filled in. You get to fill in the other two.

Matching a Graph to a Story

- A. Tom took his dog for a walk to the park. He set off slowly and then increased his pace. At the park Tom turned around and walked slowly back home.
- B. Tom rode his bike east from his home up a steep hill. After a while the slope eased off. At the top he raced down the other side.
- C. Tom went for a jog. At the end of his road he bumped into a friend and his pace slowed. When Tom left his friend he walked quickly back home.

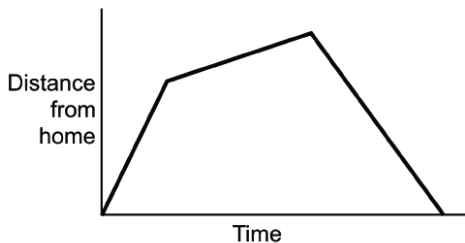
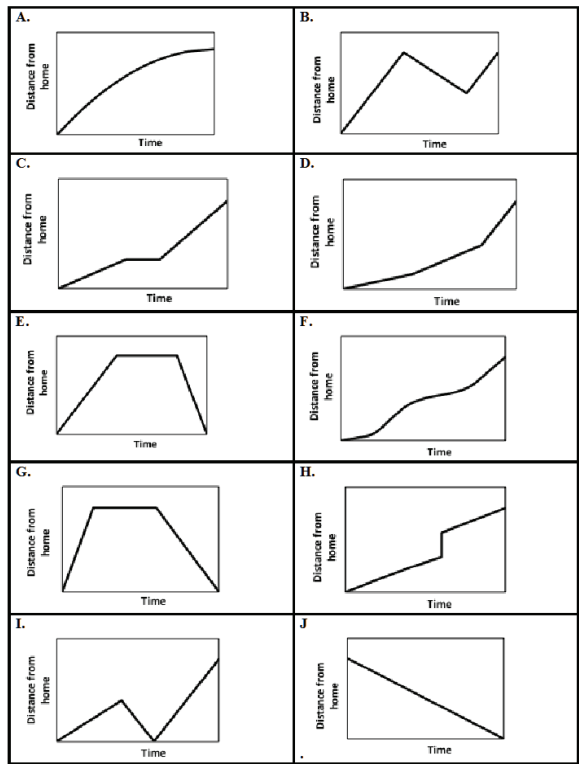


Figure 11.

Card Set A: Distance-Time Graphs



Card Set B: Interpretations

1. Tom ran from his home to the bus stop and waited. He realized that he had missed the bus so he walked home.	2. Opposite Tom's home is a hill. Tom climbed slowly up the hill, walked across the top and then ran quickly down the other side.
3. Tom skateboarded from his house, gradually building up speed. He slowed down to avoid some rough ground, but then speeded up again.	4. Tom walked slowly along the road, stopped to look at his watch, realized he was late, then started running.
5. Tom left his home for a run, but he was unfit and gradually came to a stop!	6. Tom walked to the store at the end of his street, bought a newspaper, then ran all the way back.
7. Tom went out for a walk with some friends when he suddenly realised he had left his wallet behind. He ran home to get it and then had to run to catch up with the others.	8. This graph is just plain wrong. How can Tom be in two places at once?
9. After the party, Tom walked slowly all the way home.	10. Make up your own story!

Figure 12.

Then, you get to modify your posters so that they have ten sets of matching stories, tables, and graphs. We'll put the posters up on the wall and discuss them." The discussions, moderated by the teacher, both allow the students to explain their reasoning (once again, a critically important mathematical practice) and provide the teacher more opportunities to do mid-course corrections.

That's one Formative Assessment Lesson. Another that I like a lot is "Evaluating statements about length and area." I'm just going to race through a few slides, to give you a feel for the lesson. Here's a statement: "If you draw two shapes, the one with greater area will also have a bigger perimeter." Is that sometimes, always or never true? Explain why. Note that "always" and "never" require proofs; "sometimes" requires one positive and one negative example. The lesson contains a whole bunch of these. Suppose you join the midpoints of a trapezoid. Do you sometimes, always or never divide it into two equal parts? There are three different ways to put a rectangle around a triangle, with one side of the rectangle lining up with one side of the triangle. Are the rectangles that you get sometimes, always, or never the same area? If you cut something off of a piece of a region, do you sometimes, always, or never decrease the area? Do you sometimes, always, or never decrease the perimeter? And so on, with loads of questions.

In the next phase of the lesson, we start by having students work through this problem: "If you draw the two diagonals of a quadrilateral, do you sometimes, always, or never divide the quadrilateral into four equal areas?" After they've made sense of the problem, their role shifts. Now they're given some samples of work done by other students, and told that their job is to act like a tutor: "Here is the work done by another student. Your job is to help that student. If the student is on the right track, explain how to pursue what he or she is doing so that his or her work is complete, coherent, and correct. If the student has made one or more errors, show where the work isn't right and then make a suggestion to fix what's wrong." Note that this part of the lesson focuses on another central practice in the Common Core standards: produce and critique extended chains of reason. It provides lots of opportunities to do that.

Just to give you the sense of another lesson, here's one that has aspects of modeling. This lesson, "counting trees" (Figure 13), is an estimation task. There's a picture of a tree farm in which the circles represent old trees and the triangles represent new trees. Your job is to come up with a reasonable way to figure out roughly how many circles and how many triangles there are in that picture, and say how confident you are in the answer (i.e., give upper and lower bounds for your estimate).

These lessons are built by a team of superb designers headed by Malcolm Swan at the Shell Center in Nottingham. Our goal in producing them is to provide opportunities for students to grapple with the core content and practices in the Common Core State Standards, and to support teachers in engaging in the kind of classroom activities that will help the kids to exercise their mathematical muscles and build the understandings we want them to build. In short, we want to support formative assessment—and we want to do so in curriculum-embeddable ways. Each of the lessons is targeted to one major content area at one grade in the Common Core standards. In their ideal use, the idea is to embed them into the curriculum at propitious times. Suppose, for example, you're teaching the unit on distance-time graphs. Take our lesson on distance-time graphs and put it two-thirds of the way through your unit. That way you can find out what your kids

Counting Trees

The diagram shows some trees in a tree farm.



The circles ● show old trees and the triangles ▲ show young trees.

Tom wants to know how many trees there are of each type, but says it would take too long counting them all, one by one.

Figure 13.

have made sense of, and where they're having trouble, while you still have the remaining third of the unit to do something about what you find. Thus these lessons are intended to be curriculum-independent, but at the same time, curriculum-embeddable. We went to the Gates Foundation, and said "This is what we want to do." They said, "of course" . . . after fourteen months of negotiation. In any case, we're building a hundred Formative Assessment Lessons. A main point is that our goal isn't to charge for them. This isn't about making money. We want them to be accessible. All you have to do is Google 'Mathematics Assessment' and the Mathematics Assessment Project comes up number one. There are now about sixty lessons up on the site. Within a year or so, there will be a hundred lessons—all downloadable for free, for non-commercial use.

In sum, if we do it right, the Common Core standards are going to offer us some major opportunities to teach in ways that support sense making in the classroom. That leads to the question, what does a sense making classroom look like? That's the last part of this talk. Here I'm just going to introduce the ideas; in this evening's talk I'll elaborate on them. What follows in the next few minutes is a hint of the kind of thing that I've spent the past half dozen years or so working on. The question is: "What are the attributes of powerful mathematics classrooms?"—meaning, classrooms that produce kids who are powerful mathematical thinkers. To give you a concrete image to think about, I'm going to show the video for just a couple of minutes. This comes from a book called *Connecting Mathematical Ideas* (2005) by Jo Boaler and Cathy Humphreys. Cathy is the teacher in the tape. She has the class address the following question: "I've got a 10-by-10 grid. Just in your head—no pencil, no paper—can you figure out how many squares there are on the border of the grid, how many red squares?"

[The video was played. Here is a brief summary.] Humphreys shows her class a 10-by-10 grid, with the squares on the border of the grid colored in red. She asks the students to figure out, mentally (without writing anything down) how many red squares there are. Hands go up one by one, indicating the students have an answer. She has the students talk with their neighbors about how they got their answer, and then opens things up for a whole class discussion.

Humphreys starts the whole class discussion by asking if students want to call out the answer. Yes, they say, and there is a chorus of "thirty-six." Then she asks, "Did anyone think it might be 40?" Some students say yes, and she asks why they thought that. A student explains, saying that she then realized that some squares had been counted twice. Humphreys asks if any students had thought the answer was 38, and a similar conversation takes place—with no negative consequences for the students having made a mistake. (This, of course, is what enabled them to speak freely about their thinking.) She then calls on students to explain how they arrived at 36. When things get complicated, students go up to the front of the class and make use of the diagram, explaining how they thought about the size of the border. Humphreys, meanwhile, is writing down their computations and labeling them as "A's method," "B's method," etc. The students are encouraged to explain their reasoning; Humphreys checks in with the class, saying "How many of you understood A's method?", etc. [The video was stopped here].

There is a lot more, about which we'll talk this evening. Number one, note her use of formative assessment—she was finding out what the kids understood. Number two, it was okay to be wrong in this class. "How many of you got forty? What did you do when you got forty?" The message was, "No one gets math right all the time. And it's not a sin in this classroom."

What you didn't see, but you'll see more of tonight, is that six different kids come up with six different solutions. The teacher abstracts all of them, and refers to them by name—"A's method," "B's method," etc. In the next part of the lesson she asks students to think about the border of a 6-by-6 grid: "Figure out the number of squares in the border, *using A's method*. Then using B's method, and C's, etc." You can imagine where this is going. Down the road the students will be considering the border of an n -by- n square. They'll get different algebraic expressions, such as $[4n - 4]$, $[(2n) + 2(n - 2)]$, $[(n) + 2(n - 1) + (n - 2)]$, and so on. These all look different, but shouldn't they all yield the same result? That's an introduction to some real mathematical sense making. So with that as an image for you to keep in mind, I'll skip five years of hard work and summarize the end product. In what follows I'll offer a structure for examining some of what makes for productive classrooms. What follows is telegraphic; I'll say much more about this tonight.

The big question for me is: "How can you distill what's essential in a math class down to a manageable number of dimensions?" That's where I'm ending this talk, and where I'll pick up this evening.

Here are five questions for math classes:

1. What were the big ideas and how did they get developed?
2. Did the students engage in productive struggle or was the math dumbed down to the point where they didn't get to engage productively with the mathematics?
3. Who had the opportunity to engage: just a few or everybody?
4. Who got a chance to actually speak mathematics and develop a mathematical identity?
5. Was the lesson pitched at a level where the teacher actually heard what the kids were saying and could adapt appropriately?

Those five dimensions reflect the things we want to see happening in math classes.

- Re (1): In the language of CCSSM, that's the dimension of mathematical focus and coherence, or sense making if you will. We'll take mathematical focus, coherence, and sense making as given. If they're not happening, the class is sunk.
- Re (2): The formal name of this dimension is "cognitive demand." What's the first thing that happens when a kid goes, "Teach, I don't get it."? The literature says that the most frequent response from teachers is "Here, do it this way." Well, the kid's just been deprived of the opportunity to do mathematical thinking. The question is: "Can you scaffold the kid into the right space?"
- Re (3): Who had the opportunity to engage? If the teacher always calls on the same three kids because the teacher knows they'll say the right things and help the lesson advance, then $(n-3)$ students are mathematically deprived. I'll say more about this in the next dimension.
- Re (4): Who gets to talk mathematics? Is math something to be ingested, in "demonstrate and practice" mode? Or is math something that you can make sense of, talk about, be recognized for the fact that you understand and can explain it—and in doing so, build an identity for yourself as a doer of mathematics?
- Re (5): Simply put, what role does formative assessment play?

Those are the five key questions. You can put them together in a formal rubric with a three-point scale that can be used for assessing the quality of lessons. You can develop rubrics for different classroom activity structures—for whole class, small group, student presentations. Put that all together and you get something that we very cleverly call the **T**eaching for a **R**obust **U**nderstanding of **M**athematics or the TRU Math classroom analytic scheme. The outline of TRU Math is given in Table 2; the full version will soon be available on the Algebra Teaching Study web site, <http://ats.berkeley.edu/>.

I'll explain this evening why it's important to have such a thing. But I want to stress that what I'm *not* interested in doing is grading teachers. This kind of analytic scheme is essential for purposes of research. That is, it's one thing to believe that these five dimensions are important, and it's something else altogether to demonstrate that they are. To do that you need robust empirical evidence. The way you get such evidence is to build a scheme, go into classrooms, gather data, and see whether the data indicate that the classrooms that score high on these dimensions produce students who are powerful mathematical thinkers. That's the research game. It's important, but at least as important is professional development, and along with it the questions: "How can we use these ideas? If these are the five dimensions of powerful mathematics classrooms, how can we use them as a mechanism for improving teaching?" To make the questions truly useful, we needed to expand or "unpack" them so that they provided more grist for reflecting on instruction.

We've been working on getting teachers and math coaches to use these questions as a general mechanism for planning and reflection. We're also trying to build communities of teachers who have the inclination and opportunity to visit each other's classrooms and work collaboratively on their teaching.

Imagine teacher-teacher or teacher-coach conversations that address the following before the lesson: "How am I planning for the mathematics to come across as something that really makes sense? Where am I creating opportunities for my kids to wrestle productively with the mathematics? How am I thinking about setting up the classrooms so that everybody can be productively involved? How am I thinking about giving the students an opportunity to develop senses of themselves as people who can do mathematics and communicate their understandings? How can I figure out what they're understanding, and make mid-course corrections?"

Table 2.

Episode Type		Mathematical Focus, Coherence and Accuracy	Cognitive Demand	Access	Agency: Authority and Accountability	Uses of Assessment
Whole Class Activities: Launch, Teacher Exposition, Whole Class Discussion		How accurate, coherent, and well-justified is the mathematical content?	To what extent are students supported in grappling with and making sense of mathematical concepts?	To what extent does the teacher support access to meaningful participation for all students?	To what degree are students the source of ideas and discussion of them? How are student contributions framed?	To what extent is students' mathematical thinking surfaced; to what extent does instruction build on student ideas (when potentially valuable) or address misunderstandings when they arise?
	1					
	2					
	3					
Small Group work		How accurate, coherent, and well-justified is the mathematical content?	To what extent does teacher support students to interact with their group members to make sense of mathematical concepts, or individuals by themselves?	To what extent does teacher support/ group dynamics provide access to meaningful participation and "voice" for all students?	To what extent does teacher support/ group dynamics provide access to meaningful participation and "voice" for all students?	To what degree does the teacher monitor and help students refine their thinking within small groups?
	1					
	2					
	3					
Student presentations		How accurate, coherent, and well-justified is the mathematical content?	Cognitive Demand	Access	To what degree are students the source of presented ideas and response to presented ideas?	To what degree does the teacher use student presentations to support meaningful class engagement with core ideas?
	1					
	2					
	3					
Individual work		How accurate, coherent, and well-justified is the mathematical content?	To what extent are students supported in grappling with and making sense of mathematical concepts?	Does the teacher encourage meaningful engagement from all students?	How does the teacher frame individual student contributions? To what degree do students get to propose and defend their own ideas?	To what degree does the teacher explore student thinking about a problem (whether right or wrong) and work with the student on it?
	1					
	2					
	3					

Those are “planning versions” of the questions. In reviewing the lesson, you can ask: “How did it work? And how might I do it next time?” That’s a teaser, as I said. This evening I have an expanded version of all of those in a question form we hope will actually be useful—because ultimately, the goal is to be useful. We hope to take the ideas that we’ve figured out in research and turn them into things that can help in real classrooms.

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