Journal of Mathematics Education at Teachers College

Spring – Summer 2011

A CENTURY OF LEADERSHIP IN MATHEMATICS AND ITS TEACHING

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The *Journal of Mathematics Education at Teachers College* is a publication of the Program in Mathematics and Education at Teachers College Columbia University in the City of New York.

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This issue honors Clifford B Upton who was a senior member of the Teachers College faculty from 1907 until his retirement in 1942. Professor Upton was among the Nation's most prolific mathematics authors. He served on the Board of Directors of the American Book Company enabling him to endow the Clifford Brewster Chair of Mathematics Education. The first professor to hold the Upton Chair was Dr. Myron Rosskopf.

Bruce R. Vogeli has completed 47 years as a member of the faculty of the Program in Mathematics, forty-five as a Full Professor. He assumed the Clifford Brewster Chair in 1975 upon the death of Myron Rosskopf. Like Professor Upton, Dr. Vogeli is a prolific author who has written, co-authored or edited more than two hundred texts and reference books, many of which have been translated into other languages.

This issue's cover and those of future issues will honor past and current contributors to the Teachers College Program in Mathematics. Photographs are drawn from the Teachers College archives and personal collections.

Aims and Scope

The *JMETC* is a re-creation of an earlier publication by the Teachers College Columbia University Program in Mathematics. As a peer-reviewed, semi-annual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Each issue of the *JMETC* will focus upon an educational theme. The theme planned for the 2011 Fall-Winter issue is: *Technology.*

JMETC readers are educators from pre K-12 through college and university levels, and from many different disciplines and job positions—teachers, principals, superintendents, professors of education, and other leaders in education. Articles to appear in the *JMETC* include research reports, commentaries on practice, historical analyses and responses to issues and recommendations of professional interest.

Manuscript Submission

JMETC seeks conversational manuscripts (2,500-3,000 words in length) that are insightful and helpful to mathematics educators. Articles should contain fresh information, possibly research-based, that gives practical guidance readers can use to improve practice. Examples from classroom experience are encouraged. Articles must not have been accepted for publication elsewhere. To keep the submission and review process as efficient as possible, all manuscripts may be submitted electronically at www.tc.edu/jmetc.

Abstract and keywords. All manuscripts must include an abstract with keywords. Abstracts describing the essence of the manuscript should not exceed 150 words. Authors should select keywords from the menu on the manuscript submission system so that readers can search for the article after it is published. All inquiries and materials should be submitted to Ms. Krystle Hecker at P.O. Box 210, Teachers College Columbia University, 525 W. 120th St., New York, NY 10027 or at JMETC@tc.columbia.edu

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Library of Congress Cataloging-in-Publication Data

Journal of mathematics education at Teachers College
p. cm.
Includes bibliographical references.
ISSN 2156-1397
EISSN 2156-1400
1. Mathematics—Study and teaching—United States—Periodicals
QA11.A1 J963

More Information is available online: www.tc.edu/jmetc

Call for Papers

The "theme" of the fall issue of the *Journal of Mathematics Education at Teachers College* will be *Technology*. This "call for papers" is an invitation to mathematics education professionals, especially Teachers College students, alumni and friends, to submit articles of approximately 2500-3000 words describing research, experiments, projects, innovations, or practices related to technology in mathematics education. Articles should be submitted to Ms. Krystle Hecker at JMETC@tc.columbia.edu by September 1, 2011. The fall issue's guest editor, Ms. Diane Murray, will send contributed articles to editorial panels for "blind review." Reviews will be completed by October 1, 2011, and final drafts of selected papers are to be submitted by November 1, 2011. Publication is expected in late November, 2011.

Call for Volunteers

This *Call for Volunteers* is an invitation to mathematics educators with experience in reading/writing professional papers to join the editorial/review panels for the fall 2011 and subsequent issues of *JMETC*. Reviewers are expected to complete assigned reviews no later than 3 weeks from receipt of the manuscripts in order to expedite the publication process. Reviewers are responsible for editorial suggestions, fact and citations review, and identification of similar works that may be helpful to contributors whose submissions seem appropriate for publication. Neither authors' nor reviewers' names and affiliations will be shared; however, editors'/reviewers' comments may be sent to contributors of manuscripts to guide further submissions without identifying the editor/reviewer.

If you wish to be considered for review assignments, please request a *Reviewer Information Form*. Return the completed form to Ms. Krystle Hecker at hecker@tc.edu or Teachers College Columbia University, 525 W 120th St., Box 210, New York, NY 10027.

Looking Ahead

Anticipated themes for future issues are:

Fall 2011	Technology
Spring 2012	Evaluation
Fall 2012	Equity
Spring 2013	Leadership
Fall 2013	Modeling
Spring 2014	Teaching Aids

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Journal of Mathematics Education at Teachers College
Spring–Summer 2011, Volume 2

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Design Research in the Netherlands: Introducing Logarithms Using Realistic Mathematics Education

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This article describes Realistic Mathematics Education (RME), a design theory for mathematics education proposed by Hans Freudenthal and developed over 40 years of developmental research at the Freudenthal Institute for Science and Mathematics Education in the Netherlands. Activities from a unit to develop student understanding of logarithms are used to exemplify the RME design principle of progressive formalization. Starting from contexts that elicit students' informal reasoning, a series of representations and key questions were used to build connections between informal, pre-formal and formal representations of mathematics. Student and teacher comments from the pilot of this unit in a College Algebra course at a U.S. community college suggest this approach may benefit students who have been underserved by traditional approaches to mathematics instruction.

Keywords: instructional design principles; student reasoning; problem contexts; representations; progressive formalization; realistic mathematics education; logarithms

The historical foundations for teaching school mathematics in the United States emerged during the first half of the 20th century, an era dominated by behaviorist assumptions regarding student learning. These origins established a sustained mathematical experience in the United States that, until recently, often favored students with recall skills, a desire for precision, and a tolerance for repetition as practice (De Corte, Verschaffel, & Greer, 1999). Left behind, unfortunately, have been a multitude of students who are nonetheless capable of doing mathematics. Often those students who struggle with higher mathematics desire opportunities to make sense of mathematical relationships, which are the foundation of a more robust understanding of mathematics.

Realistic Mathematics Education

In many mathematics classes, particularly in post-secondary developmental mathematics courses, teachers believe that students need to be told *how* to solve a problem, instead of motivating students to think for themselves. Students watching a teacher work through several examples (i.e., worked-example instruction; Sweller & Cooper, 1985) followed by individual student practice on assigned problem sets is still the principal method of instruction in U.S. mathematics classrooms. Rarely do secondary and community college students have the opportunity to explore a new topic or representation through meaningful problem contexts. Instructional approaches often focus on student training in formal mathematics without including contexts and pre-formal representations, which is counterproductive for many students who desire to make sense of the mathematics they encounter. For such students, the lack of relevance and mathematical sense making often results in frustration, disengagement and dropping the course.

During a two-week study to test the feasibility of adapted Dutch materials in U.S. community college classrooms, we collaborated on the design of a unit to promote student understanding of logarithms using the instructional design theory of Realistic Mathematics Education (RME). Originally proposed by the Dutch mathematician Hans Freudenthal in the 1970s, an essential principle of RME is that engagement in mathematics for students should begin within a meaningful context. The development of understanding and the ability to make sense of mathematical representations begins with the student's own informal reasoning, or in the words of

Freudenthal (1991), with "common sense." Drawing from a cognitive perspective of learning, students connect prior knowledge to new mathematical representations, concepts, and skills. As a result, a more robust way of knowing and doing mathematics is constructed from the student's perspective. This approach gives students a greater sense of ownership. Although the role of the teacher is essential to help students collectively negotiate the meanings and use of conventional mathematical terms, symbols, representations, and procedures. It is important to point out here that the "realistic" aspect of RME is not just because of its connection with real world contexts, but it is related to the emphasis that RME puts on offering students problem situations which are imaginable. The Dutch translation of "to imagine" is "zich realiseren," and so it is this emphasis—on making something real in your mind—that gives Realistic Mathematics Education its name (van den Heuvel-Panhuizen, 2000). Real world contexts can be used but this is not always necessary. More often, contexts are idealized to motivate powerful mathematical strategies.

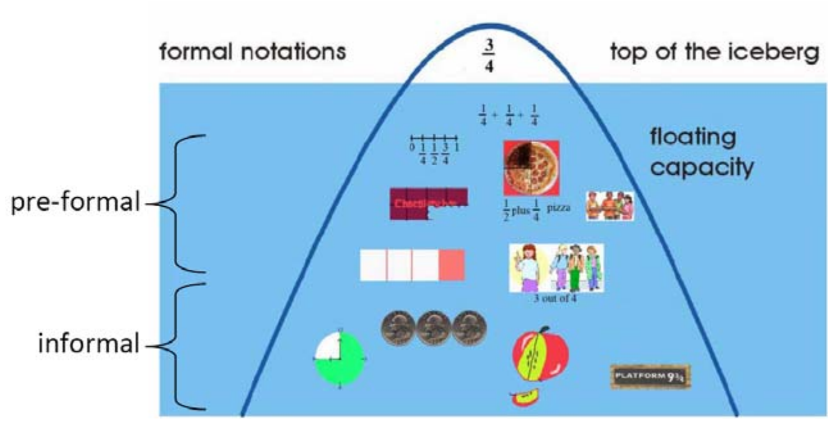


Figure 1. Iceberg for fractions

Over the past 40 years, research faculty at the Freudenthal Institute for Science and Mathematics Education (FiSME[1]) have focused on design and development research in mathematics education (Gravemeijer, 1994; de Corte, Greer, & Verschaffel, 1996). As a result, the Netherlands has sustained a successful track record in international comparisons of mathematics achievement (e.g., TIMSS and PISA). In the United States and other countries, research projects that have utilized approaches based on FiSME's research have demonstrated similar success in motivating teacher change in classroom practice in elementary (Cobb, McClain, & Gravemeijer, 2003), middle (Romberg, 2004; Her & Webb, 2004), secondary (de Lange, Romberg, Burrill, & van Reeuwijk, 1993), and college level mathematics (Rasmussen & King, 2000; Larsen & Zandieh, 2007), with resulting gains in student achievement (Romberg & Shafer, 2008). To exemplify the design principles used for RME curriculum, instruction and assessment, this article focuses on *Exponents and Logarithms*, a unit developed for U.S. community college students enrolled in College Algebra (based on the original unit *Exponenten en Logaritmen*, designed by Jan de Lange [1978]).

Progressive Formalization

At the heart of RME is the didactical construct of progressive formalization. The "process of going from the concrete to the abstract" has been espoused in education psychology literature in the United States for at least a century (cf. Dewey, 1910) and can be traced back to 18th century education-related writings in Europe (e.g., Rousseau, Pestalozzi, Froebel, Montessori, etc.). Yet, RME offers more than a way to support student transition from the concrete to the abstract. RME instructional sequences are conceived as "learning lines" in which problem contexts are used as starting points to elicit students' informal reasoning. That is, the context is a source for new mathematics.

When appropriate, the teacher introduces students to pre-formal strategies and visual models that are progressively more formal to support their mathematical sense-making. Pre-formal strategies are often more abbreviated and efficient (e.g., the use of "chunking" of larger values when solving a division problem rather than using repeated subtraction or directly counting members of a group). Pre-formal models are representations that can be used to solve problems across various contexts, such as a ratio table or a double number line to solve a proportion. Pre-formal strategies and models offer the additional benefit of being more closely related to how a student reasons about a problem. Sometimes not as efficient as a formal algorithm, pre-formal strategies and models are often better understood. Treffers (1987) described this as "horizontal" and "vertical" mathematization, where horizontal mathematization is the process of developing mathematical tools to solve a problem in a realistic context and vertical mathematization is advancing within mathematical domains.

Problem contexts, visual representations, pre-formal strategies and formal mathematics are intertwined in RME curriculum and instruction. Contexts are not added at the end of the learning line, where they are often excluded or perceived as an afterthought. Instead, contexts serve as realistic starting points to support student engagement and elicit student thinking. These contexts are then coupled with successive problems, representations, and strategies

[1] http://www.fi.uu.nl/en/

to develop a coherent learning line that continues to build and strengthen the connections between contexts, concepts, procedural knowledge, and student understanding of formal mathematics. To convey how progressive formalization is a requisite principle in the design of instructional sequences, FiSME researchers developed the "Iceberg Model" to illustrate how informal, pre-formal and formal mathematical models and strategies are used by students to develop a "floating capacity" for the understanding of formal representations of mathematics (see Figure 1).

The iceberg consists of the "tip of the iceberg" and a much larger area underneath, which is designated as the "floating capacity." The tip of the iceberg represents the formal procedure or symbolic representation of interest. However, before this formal level is reached, skills and insights at a less formal level need to be elicited from students and developed (floating capacity). Using this approach, students' prior knowledge is assessed through their responses to realistic contexts, which motivate the use of mathematical language. Later, students use structured models, which lead to a deeper understanding of more symbolic, formal representations (Boswinkel & Moerlands, 2001; Webb, Boswinkel & Dekker, 2008).

This model has been an effective way to communicate that starting with formal procedures and ignoring the meaningful representations below the surface is not the most effective way to facilitate student understanding of mathematics. The direct-formal approach, in the absence of other well-known representations (e.g., fraction bars, arrays, number line, etc.), encourages students primarily to use recall as their approach. In contrast, taking advantage of less formal representations may have greater potential for relating students' informal knowledge, promoting number sense, facilitating student problem solving, and establishing representational connections which lead to a deeper understanding of mathematics.

Using Progressive Formalization to Introduce Logarithms

In most textbooks, logarithms are introduced as the inverse of exponentials ($y = b^x \Leftrightarrow x = \log_b y$). Using formal notations of exponential rules, logarithmic rules (e.g., log *a* + log *b* = log *ab*) are derived. Although mathematically correct, this approach does not offer students a meaningful basis to relate logarithms to more familiar mathematical representations. To conceptualize logarithms, students need to understand exponential growth.

Using the RME approach, a lesson series on logarithms initially developed for use in the Netherlands (de Lange, 1978), was translated and adapted for use as a replacement chapter in a College Algebra course at a community college. The lesson series had three parts. Part A addressed exponential processes in real life contexts. Part B focused on formal exponential functions and the rules of exponents. In part C the logarithm was introduced using the context of time and growth.

On the same day two friends both buy a foal. Both foals have a weight of 50 kilogram. After one month they compare the weights.

Fred says, "My foal grew 10 kg." Andy answers, "My foal grew 20 %".

After another month they meet again and compare the weights.
Fred: "another 10 kg!"; Andy: "another 20%."

A1 What are the weights of the foals **two months** after they were bought?

Figure 2. Introduction to Unit

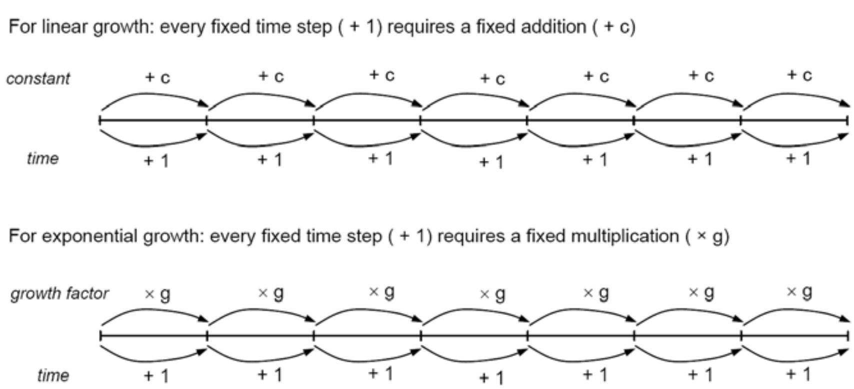


Figure 3. Visualizing Characteristics

Informal and Pre-Formal

The concept of exponential growth was presented by contrasting linear and exponential growth in a quasi-realistic context:

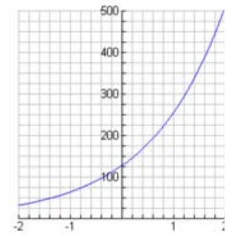
The initial context and question in Figure 2 addresses a misconception about percentages and is an opportunity to contrast exponential and linear growth. In the class, discussions about the different growth patterns led to the characteristics of linear and exponential growth, which served later as a pre-formal model for further reasoning about exponentials and logarithms (Figure 3). Although this use of the double number line may seem misleading, the similarities between the two visuals (i.e., constant additive and constant multiplicative growth) establish the relationship between the slope of linear functions and the base for exponential functions. In both cases, the duration of the growth period, *t*, will be a particular variable of interest.

Additional examples of problems that relate informal and pre-formal reasoning include an investigation of bacteria that is reported to double in volume every two hours (Figure 4). The choice of doubling as a growth factor for *E. coli* over a given time period is intentional in order to generate a pattern that is easier for students to work with than the real growth factor. The choice of doubling as growth factor also plays a role when the concept of a logarithm is introduced. In addition, the use of the graph for different volumes and related ratios emphasizes the stability and predictability of function when the difference between two time periods is the same.

E.coli:

Starting with 128 E.coli bacteria that double in volume (and number) every two hours, the function that describes this growth process is: $V(t) = 128 \cdot 2^{t}$ with t the time (in units of 2 hours) and V the volume of E.coli.

At $t = 0$, the volume of E.coli is 128. At $t = 1$, the volume (and also the number) is doubled to 256. But between $t = 0$ and $t = 1$, the volume is exponentially increasing from 128 to 256. As you can see, the continuous graph shows all values in between 128 and 256.



A14 **a** Using the graph, make an accurate estimate of the volume of E.coli after one hour (at the moment $t = 0.5$) and also after 3 hours ($t = 1.5$).
b Dividing both numbers, the result should be exactly 2. Why?
c Use your graphing calculator to find the values for $V(0.5)$ and $V(1.5)$.
d Explain why $\frac{V(0.5)}{V(0)}$, $\frac{V(1)}{V(0.5)}$, $\frac{V(1.5)}{V(1)}$ and $\frac{V(2)}{V(1.5)}$ should all have the same outcome.
e What is the outcome for each ratio? Explain what this outcome means.
f What does this outcome have to do with the growth factor of 2 (for two hours)?

Figure 4. Growth of E.coli

The Switch to Logarithms

After working with exponentials, the first problem in the logarithms section introduces a context that requires analysis of graphs and tables for the area covered by a water plant (duckweed) that doubles in size every day (Figure 5). Students are asked to interpret a graph to determine the time it takes for the water plant to grow to x square meters. This becomes an extension of the foal and bacteria problems. While completing a table with values that can be read from the graph, students are asked to find times for areas that cannot be found in the graph.

A table that summarizes many different times and the respective areas for duckweed is then presented to students (Figure 6). Students are asked to find patterns in the table and explain why these patterns make sense. Using the notation $t(A)$ as an abbreviation of "the time (t) it takes to cover area (A) when starting with an area of 1 square meter," $t(8) = 3$ means that it takes 3 days for the area to grow to 8 m^2, or eightfold its original area. Likewise, $t(20) = 4.32$ means that it takes 4.32 days for the original 1 m^2 to grow to 20 m^2.

Using the table, students find patterns like: $t(5) + t(3) = t(15)$, $t(25) = 2\cdot t(5)$ and $t(18) - t(2) = t(9)$. The patterns make sense to students and are pre-formal precursors to the product, power, and quotient rules for logarithms.

Formalization

Between the sections on exponents and logarithms, context-oriented problems are progressively decreased while increasing the formal representation of exponential growth and decay. Visual representations, such as the double number line, are used to study mathematical structure and generalizable aspects of exponential phenomena. Yet during the discussion of the rules for exponents and related concepts, students often refer back to the previously explored contexts as a referent for making sense of more formal representations. Further generalization of this relationship eventually leads students to propose the product rule. Depending on the student's varying abilities to reason with abstract symbolic representations, some will reason with logarithms without any reference to context while others will refer back to contexts involving time.

For example, the expression $\log_2 (5) + 3$ can be considered within the context of time-area as the time it takes to quintuple in area plus three more days (during which the area grows 8 times, 2x2x2), resulting in the time it takes to reach 40 times the original area. The formal representation of this situation is:

$$\log_2 (5) + 3 = \log_2 (5) + \log_2 (2^3) = \log_2 (5 \times 8) = \log_2 (40)$$

Even for those students who are comfortable with symbolic representations, this instructional approach provides them with conceptual referents that are generative and could be applied when they encounter new problem situations.

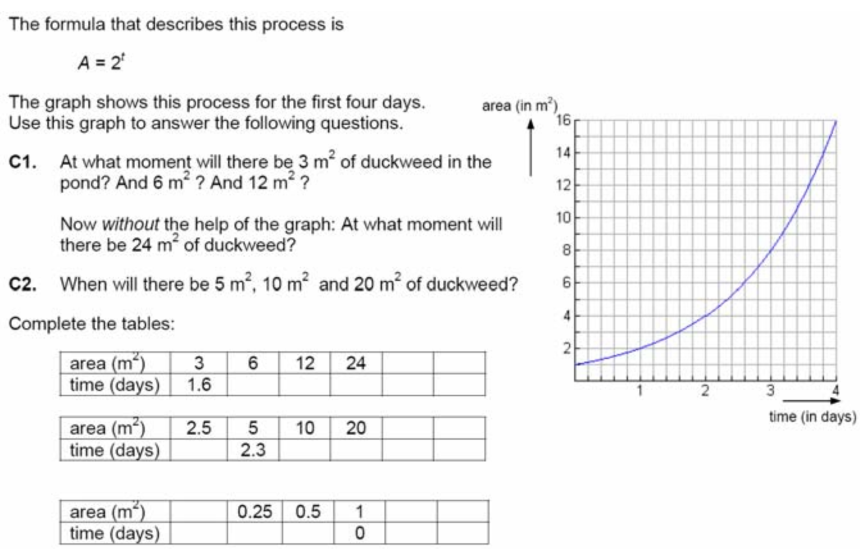
Results from the College Algebra Two-week Pilot

The results presented here are used to illustrate the reactions from community college students (who were generally unsuccessful in their prior experiences with high school algebra) and Monica Geist (who had over 10 years experience teaching College Algebra). According to Geist, students were engaged at a higher level than at any other time in the course, or in any other mathematics course that she had taught. With so much emphasis on realistic examples and visual representations, students had something to "hook" their new knowledge onto. When contemplating how logarithms were previously taught, Geist realized that students did not have a schema for understanding logarithms. For students from previous semesters, it seemed that logarithms were a set of random rules that had to be memorized in order to pass the test. In response to the RME unit, Geist noted that students were able to propose more complex and related mathematical structures and offer generalizations of mathematical relationships. Students were motivated to ask timely, unsolicited questions that promoted further discussion of complex mathematical structures and proposed the general rules for exponential and logarithmic functions before they were presented explicitly in class. The two-week trial demonstrated that students could acquire a profound understanding (Ma, 1999) of the connection between exponential growth and logarithms.

The time invested in developing student understanding of exponential growth was the key to their understanding of logarithms. This principle of using realistic contexts to promote student understanding is rarely found in units on logarithms. Typically, the format of algebra textbooks introduces concepts and principles first with applications problems presented *later*, assuming there is enough time to get to the application problems. For this community college instructor, the level of student engagement and the questions students asked convinced her that this approach was a significant improvement over the previous approach. To her, the most compelling evidence of success was the mathematical insights shared by her students. The following are a selection of students' reflections about their experiences with logarithms, previously "the most misunderstood concept of the entire College Algebra course":

- I had to figure it out for myself, so it stuck in my brain.
- I understood the main idea. It wasn't just performing problems, it was understanding the *why* behind the problem. It used to be if you put a problem in front of me, I couldn't do it unless you told me what you wanted me to do. Here I understand everything and can just do it.
- It taught me more than just how to "do" the problems—I understand it! Loved the experience, it was great! (P.S. You should really do every chapter like this. *I'm serious!*)
- I came out of this with some understanding and I think under different circumstances that may not be the case. The visual charts and graphs were very helpful as was the fact that we were able to use both words and math on the test to get our points across.

It is noteworthy that the response of many students in the end-of-unit evaluation survey revealed an appreciation of learning and understanding mathematics, and not merely an interest in earning a passing grade. Their feedback indicated the viability of this approach for increasing the students' understanding of algebra.


The formula that describes this process is

$A = 2^t$

The graph shows this process for the first four days. Use this graph to answer the following questions.

C1. At what moment will there be 3 m^2 of duckweed in the pond? And 6 m^2? And 12 m^2?

Now *without* the help of the graph: At what moment will there be 24 m^2 of duckweed?

C2. When will there be 5 m^2, 10 m^2 and 20 m^2 of duckweed?

Complete the tables:

area (m^2)	3	6	12	24		
time (days)	1.6					

area (m^2)	2.5	5	10	20		
time (days)		2.3				

area (m^2)		0.25	0.5	1		
time (days)				0		

Figure 5. Graph of Duckweed

A	1	2	3	4	5	6	7	8	9	10	11	12	13
t	0	1	1.58	2	2.32	2.58	2.81	3	3.17	3.32	3.46	3.58	3.70
A	14	15	16	17	18	19	20	21	22	23	24	25	26
t	3.81	3.91	4	4.09	4.17	4.25	4.32	4.39	4.46	4.52	4.58	4.64	4.70

Figure 6. Table of Area vs. Time

No Rewards without Risks

From this example and the other design experiments that were previously noted, Realistic Mathematics Education and the principle of progressive formalization serve as generative design principles for both developing instructional materials and informing instructional decisions. Teachers who choose to use this approach need to understand that contexts can initiate the investigation of new mathematics, and that pre-formal models and strategies should be welcomed for promoting the understanding of formal mathematical goals (Webb, 2008).

Implementation of such materials is not trivial. It demands a deeper understanding of the targeted mathematics topics and a greater relational understanding (Skemp, 1978) of how contexts and representations can be used to elicit student reasoning and promote learning (e.g., Webb, 2010). Using unfamiliar and innovative materials requires an instructor to take risks with their students, since this was most likely not the way the instructor learned mathematics. The co-authors of this paper worked to design lessons and to support the instructor in distributing the risk (Webb, Romberg, Burrill & Ford, 2005). In spite of uncertainty, the instructor's commitment to implement these materials was rewarded by the students' tangible success when faced with challenging content and the instructor's own motivation to adapt other courses by using RME and progressive formalization. This subsequent adaptation of additional courses by Geist and, soon after, by other instructors in her department (Geist, Webb & van der Kooij, 2008) attests to the utility of RME in informing the design of a more student-centered mathematics classroom.

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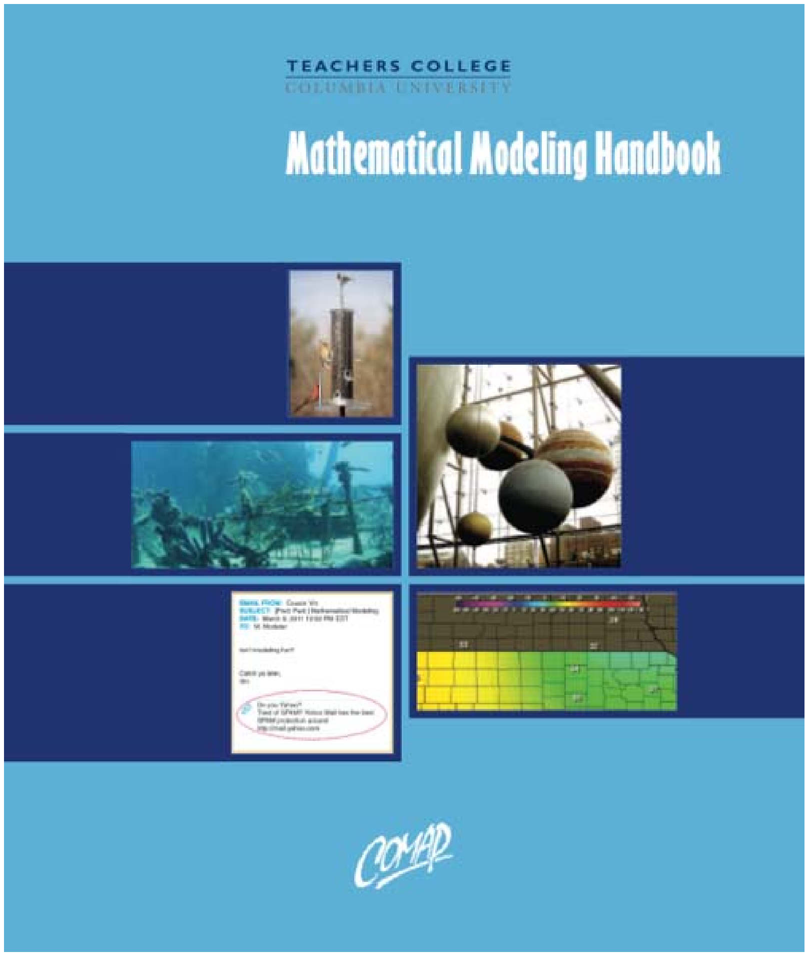

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