Journal of Mathematics Education at Teachers College

Fall – Winter 2011

A CENTURY OF LEADERSHIP IN MATHEMATICS AND ITS TEACHING

The *Journal of Mathematics Education at Teachers College* is a publication of the Program in Mathematics and Education at Teachers College Columbia University in the City of New York.

Dr. Robert Taylor was selected by the Teachers College sponsored Teachers for East Africa program to teach mathematics of Uganda's Makerere University. He returned to TC as an instructor in the Department of Mathematics, Statistics, and Computing in Education where he developed an innovative programming language (FPL) intended to introduce educators to the then-new field of computer programming. His seminal work entitled *Computers: Tutor, Tool, Tutee* led to leadership in the new field of computers in education. Dr. Taylor completed 33 years as a member of the Teachers College faculty in 2009.

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Aims and Scope
The *JMETC* is a re-creation of an earlier publication by the Teachers College Columbia University Program in Mathematics. As a peer-reviewed, semi-annual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Each issue of the *JMETC* will focus upon an educational theme. The themes planned for the 2012 Spring-Summer and 2012 Fall-Winter issues are: *Evaluation* and *Equity*, respectively.

JMETC readers are educators from pre K-12 through college and university levels, and from many different disciplines and job positions—teachers, principals, superintendents, professors of education, and other leaders in education. Articles to appear in the *JMETC* include research reports, commentaries on practice, historical analyses and responses to issues and recommendations of professional interest.

Manuscript Submission
JMETC seeks conversational manuscripts (2,500-3,000 words in length) that are insightful and helpful to mathematics educators. Articles should contain fresh information, possibly research-based, that gives practical guidance readers can use to improve practice. Examples from classroom experience are encouraged. Articles must not have been accepted for publication elsewhere. To keep the submission and review process as efficient as possible, all manuscripts may be submitted electronically at www.tc.edu/jmetc.

Abstract and keywords. All manuscripts must include an abstract with keywords. Abstracts describing the essence of the manuscript should not exceed 150 words. Authors should select key words from the menu on the manuscript submission system so that readers can search for the article after it is published. All inquiries and materials should be submitted to Ms. Krystle Hecker at P.O. Box 210, Teachers College Columbia University, 525 W. 120th St., New York, NY 10027 or at JMETC@tc.columbia.edu

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Journal of Mathematics Education at Teachers College
Fall–Winter 2011, Volume 2

The Mathematics of Global Change

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This paper is a descriptive and preliminary report on recent efforts to address two questions: 1) Can school mathematics be used to enhance our students' ability to understand their changing world? and 2) What role might computer technology play in this regard? After recounting some of the mathematical tools that led to a better understanding of celestial change, an analogous approach is used to address terrestrial change here on earth. This involves an incisive look at Fibonacci's rabbit problem and leads to the consideration of "a Copernican metaphor," one in which efforts to address environmental issues are related to the transition from a geocentric to heliocentric model of the solar system.

Keywords: Computer technology, celestial change, terrestrial change, Fibonacci, solar system.

Introduction

Amid the pomp and splendor of a Harvard graduation, the question "Why is it hotter in summer than winter?" was posed to 23 individuals—students, faculty, and alumni. Only two were able to answer correctly, and a remarkable video (still available online) presents some of the animated but incorrect efforts at explanation. This video has been used to support establishment of "The Private Universe Project" at Harvard's Center for Astrophysics as well as a number of science education reform programs supported by NSF, the Annenberg Foundation, and others.

The apparent lack of interest by the mathematics education community in such matters strikes one as curious. School mathematics has an important role to play in conveying an understanding of phenomena such as seasonal change, while topics from astronomy provide a wealth of engaging applications. Shouldn't the use of mathematics in understanding the world about us figure prominently into mathematics curricula?

Such questions were brought to the fore in October, 2010 when Terence Tao delivered the American Mathematical Society's annual Einstein Lecture. Rather than speaking about his work in one of the many areas to which he has contributed, Tao used much of this occasion to describe the role of elementary mathematics in developing a sense of scale for the earth, our solar system, and the cosmos (see Figure 1). He also discussed the role that such numeracy played in arriving at an understanding of celestial change, notably Kepler's laws. Under the heading "The Cosmic Distance Ladder," much of Tao's presentation was accessible to students conversant with basic geometry and proportional reasoning. Inherent interest aside, Tao's lecture also provides a point of departure for some rather different efforts to relate school mathematics to issues of global change. What follows is an account of one such effort aimed at students at grades 8-12.

The California State Summer School for Mathematics and Science, better known as COSMOS, is a four-week summer program for talented students that is offered at four campuses of the University of California. "The Mathematics of Global Change" is the title of a 2-week course that was offered in 2011 at UC Davis. Part of this course was presented in a rather traditional classroom format. This was, however, reinforced by a variety of computer lab projects that made use of spreadsheets, dynamic geometry software, and TI-83 calculators.

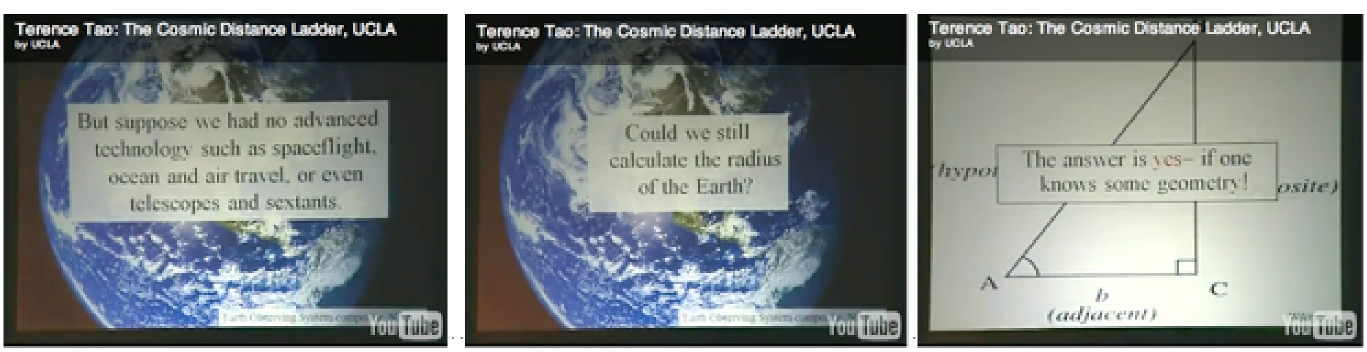
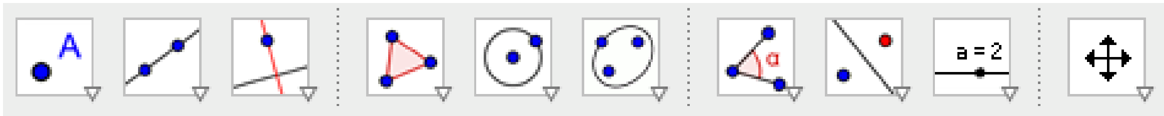


Figure 1



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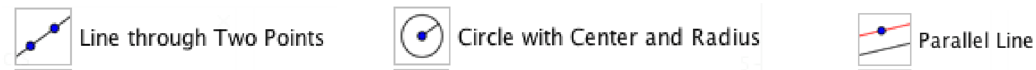

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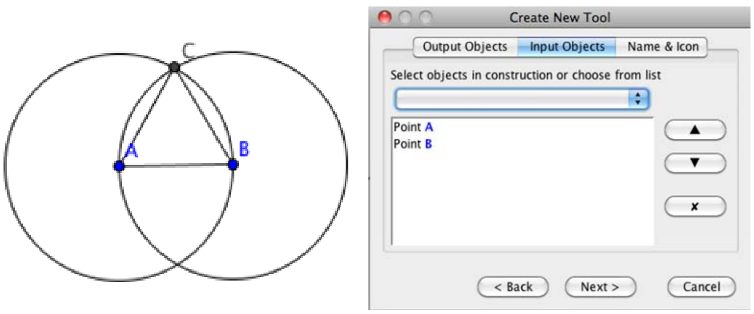

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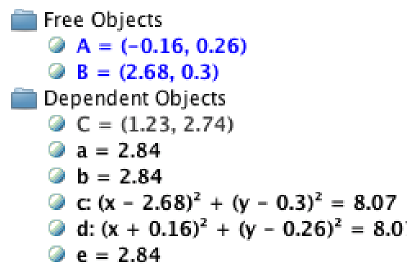

Figure 5

Geometry

Tao began his ascent of the cosmic distance ladder by recounting Eratosthenes' measurement of the size of the earth.[1] Noting that the ancient Greeks lacked tools that we tend to take for granted, he asserted that such difficulties can be overcome "if one knows some geometry!"

Our COSMOS course began by acquainting students with Book One of Euclid's *Elements*, a work rather different from the Euclidean geometry they encounter in the standard school curriculum. Even though Euclid's synthetic geometry is devoid of numerical measurement, it was crucial to the first few steps up the cosmic distance ladder.

Dynamic geometry software provides an engaging way of conveying the spirit and nature of synthetic geometry. In the case of the freeware program GeoGebra, the user is offered a myriad of tools (Figure 2), only three of which are needed to bring Euclid's five postulates to bear (Figure 3). With these tools one can emulate the beginnings of Book One by constructing an equilateral triangle with given base AB. But rather than regarding this construction a truth to be referred to in future deductions, GeoGebra allows us to save it as a new tool (Figure 4).

In using GeoGebra in this way, students must exercise restraint lest they make use of the other tools that such technology offers up so conveniently[2]. They must also suppress the Algebra View option which offers to display the Cartesian coordinates that GeoGebra assigns to points, the lengths of line segments created, equations associated with the constructed circles, etc. (Figure 5).

In this Euclidean framework, problems such as "construct a regular pentagon with given side AB" and "dissect a rectangle ABCD so that the pieces can be rearranged to form a square" convey both the spirit and the power of synthetic geometry.

The Cosmic Distance Ladder

Our account of "Eratosthenes measures the size of the earth," began by recalling the existence of a well in Syene (now Aswan) with the property that at noon on midsummer day, the sun shines directly to the bottom of the well. In Alexandria, directly north of Syene, Eratosthenes found that the sun's rays make an angle of 7.5° with the vertical. Proposition 29 of Book 1 (now paraphrased as *alternate interior angles are equal*) enabled him to conclude that angle AOS is also 7.5 degrees (Figure 6). This leads to the

[1] In his 10/10/2010 blog, Tao states "I am happy to have the text or layout of these [PowerPoint] slides used as the basis for other presentations, so long as the source is acknowledged."

[2] Unfortunately, there does not seem to exist a form of dynamic geometry that provides the teacher with control over the features available to the student user.

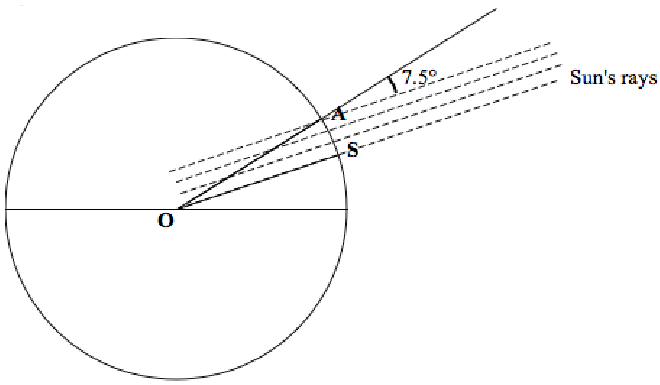

Figure 6

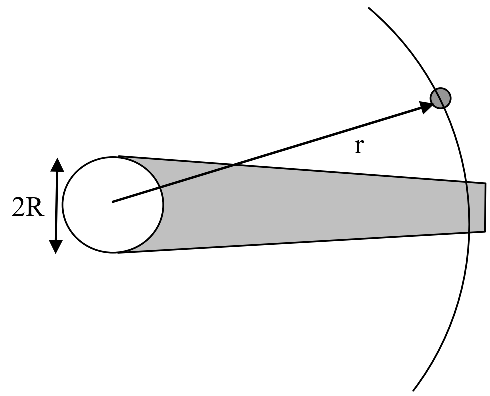

Figure 7

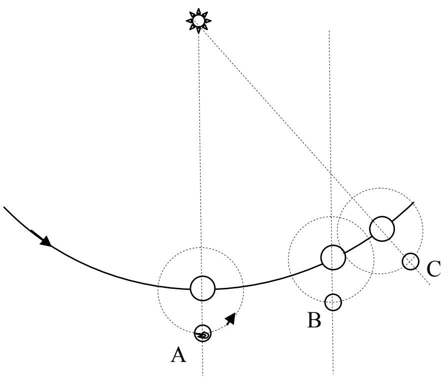

Figure 8

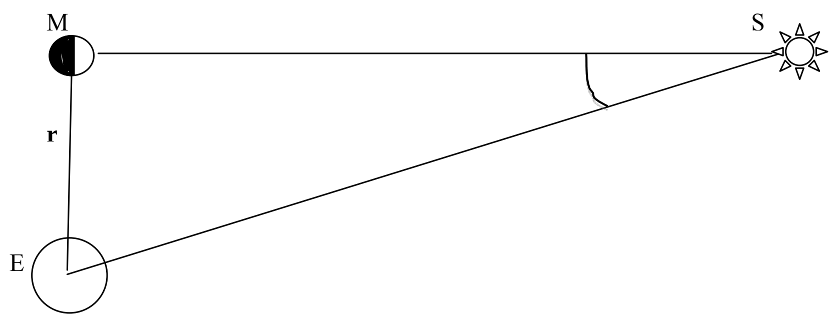

Figure 9

proportion 360:7.5 ≈ Circumference:AS and the conclusion that the circumference of the earth is 48 times the length of the arc AS. Knowing that AS is 500 miles (5000 stadia), Eratosthenes arrived at the very credible estimate of 24,000 miles for the earth's circumference.

With such a measure in hand, Aristarchus was able to use lunar eclipses to estimate the distance from earth to the moon (Figure 7). Assuming the earth's shadow to be cylindrical, noting that the duration of a lunar eclipse is at most 3 hours, and taking the lunar month to be 29.5 days (708 hours), one arrives at the proportion $2\pi r:2R = 708:3$ and the estimate $r \approx 75R$. While this is the right order of magnitude, arriving at Tao's better estimate of $r \approx 60R$ requires several refinements. One of these concerns is the duration of the lunar month, which earthlings find it natural to define as the duration between full moons. This *synodic* lunar month is represented by configurations A and C in Figure 8.

However, a remote observer would assert that a complete lunar revolution about the earth occurs earlier! It is in configuration B, in Figure 8, that the line determined by the earth and moon is parallel to its position at A. This observation leads to a *sidereal* month of about 27.3 days and contributes to an improved estimate for r.

From here one is (in theory) able to estimate the distance from earth to the sun by noting that a moon that is half full signals a right triangle configuration for the earth, moon and sun (Figure 9).

Given r = 240,000 miles, an accurate measure for angle ESM would enable one to calculate ES. This step up the cosmic distance ladder leads to the analogous questions about the other planets.

It is here that Johannes Kepler enters the picture with his study of the motion of earth and Mars. Central to this effort was Kepler's use of the synodic year to calculate Mars' sidereal year. As described by Albert Einstein on the 300th anniversary of Kepler's death,

> Johannes Kepler found a marvelous way…to ascertain the real shape of Earth's orbit. Imagine a brightly shining lantern somewhere in the plane of the orbit. Assume we know that this lantern remains permanently in its place and thus forms a...fixed triangulation point for determining the Earth's orbit, a point which the inhabitants of Earth can take a sight on at any time of year. Let this lantern be further away from the Sun than the Earth. With the help of such a lantern it is possible to determine the Earth's orbit in the

> following way. There comes a moment when the Earth (E) lies exactly on the line joining the Sun (S) and the Marvelous Lantern (M). If at this moment we look from the Earth (E) at the Lantern (M) our line of sight will coincide with the line Sun-Lantern (SM). Suppose the line to be marked in the heavens. Now imagine the Earth in a different position and at a different time. Since the Sun (S) and the Lantern (M) can both be seen from the Earth, the angle at E in the triangle SEM is known. We might do this at frequent intervals during the year, each time we should get on our piece of paper a position of the Earth with a date attached to it and a certain position in relation to the permanently fixed base SM. The Earth's orbit could thereby be determined.

Having described Kepler's method of triangulation, Einstein went on:

> But, you will ask, where did Kepler get his lantern? His genius and nature gave it to him. There was the planet Mars, and the length of the Martian year was known.

By way of classroom version, the observation that successive oppositions of Mars occur every 780 days (or about $2\frac{1}{7}$ years) enabled Kepler to conclude that Mars has $1\frac{1}{7}$ (or 8/7) rotations about the sun in the time that the earth has $2\frac{1}{7}$ (or 15/7). Therefore the angular velocity of Mars is 8/15 that of earth, so that the duration of the Martian year is about 15/8 earth years. It was a more precise version of this calculation that led Kepler to the value of 687 days for the duration of the Mars year. This enabled him to make use of a crucial fact: For data gathered 687 days apart, the sun and Mars define a fixed coordinate axis relative to which Kepler was able to locate and track a moving earth. It was these calculations that enabled him to conclude that the earth, and then Mars, move in elliptic orbits about the sun, rather than the circular orbits that Copernicus had posited.

Global Change

While Tao now continues his ascent of the cosmic distance ladder with distances to various stars and galaxies, we turn our attention back to earth and some very accessible forms of numeracy that are rarely noted. Recalling Eratosthenes's estimate of 4000 miles for the radius of the earth and Archimedes' formula $S = 4\pi R^2$ for the area of a sphere , one is led to an estimate of 200 million square miles for the surface area of the earth. Since almost ¾ of this is water and one square mile equals 640 acres, there are about 32 billion acres of land on earth. This leads to the question, "How much of the earth's land is arable?"

Since the internet offers a wide range of answers, we can consult the *CIA World Factbook* (https://www.cia.gov/library/publications/the-world-factbook). Here data on arable land is presented for every country on earth. When averaged over the entire world, it leads to an answer to two decimal places: 10.57%. Combined with our calculation of 32 billion acres of land on earth, one concludes that there are about 3.4 billion acres of arable land, or slightly less than ½ acre per person on earth.

Delayed Exponential Growth

Since such calculations raise the specter of "Malthusian thinking," it becomes important to help students move beyond the simplistic study of exponential growth, perhaps by recalling some issues raised over 800 years ago by Fibonacci's famous rabbit problem. Here we deal with a rather fecund breed of rabbits, each mating pair of which has a pair of babies each month. Starting with one such pair, Fibonacci asked for the number of pairs at the end of 12 months.

At first glance, this may suggest a simple problem of exponential growth. Letting u(n) denote the number of pairs at the end of the n-th month, we have u(0) = 1. Letting R = 1 denoting a monthly growth rate of 100%, the compound interest rule u(n) = u(n-1) + Ru(n-1) leads to $u(n) = (1+R)^n u(0)$ and

$$u(12) = 2^{12} = 4096.$$

But by way of making the problem more realistic, Fibonacci added the provisos that

1. Only adult pairs have babies
2. Babies take 1 month to mature.

These conditions lead us to write u(n) = a(n) + b(n), where a(n) and b(n) denote the number of adult pairs and baby pairs, respectively, so that u(n) - u(n-1) = a(n-1). In the absence of mortality, adult pairs in any given month will consist of the prior month's adults (all of which survived) and the prior month's babies (all of which matured). That is,

$$a(n-1) = a(n-2) + b(n-2) = u(n-2)$$

leading to the *delay* difference equation

$$(1) \quad u(n) - u(n-1) = u(n-2)$$

By stipulating that the problem starts with one pair of *babies*, Fibonacci also imposed the initial conditions u(0) = u(1) = 1. This leads to the famous Fibonacci numbers

$$\{1, 1, 2, 3, 5, 8, 13, 21, 34, ... \}$$

whose properties are often presented as a form of mathematical recreation. In fact, however, Fibonacci was calling our attention to an important fact: *Biological* *systems tend to involve delays*, a phenomenon that is often overlooked in calculus-based forms of modeling.

By way of relating this problem to exponential growth, recall that the formula u(n) = (1+R)u(n-1) enables one to recover the value of R from the sequence {u(0), u(1), u(2),...}. That is, for any n > 0 we simply calculate the ratio u(n)/u(n-1) to obtain 1+R. To calculate the growth rate of the Fibonacci numbers we use (1) to write

$$\frac{u(n)}{u(n-1)} - 1 = \frac{u(n-2)}{u(n-1)}$$

The assumption that u(n)/u(n-1) approaches x now leads to the quadratic equation

$$x - 1 = 1/x \quad \text{or} \quad x^2 - x - 1 = 0$$

with positive solution

$$\frac{1+\sqrt{5}}{2} \approx 1.618$$

	A	B	C	D	E	F	G	H	I
1	**This spreadsheet calculates the solutions of u(n) - u(n-1) = R₁u(n-1) for n ≤ 4 and of v(n) - v(n-1) = R₂v(n-1) for n ≤2. It then calculates the solution of v(n) - v(n-1) = R₂v(n-3) for n > 2 and the ratios u(n)/u(n-1) and v(n)/v(n-1).**								
2									
3	R_1 =	0.1	R_2 =	0.12					
4									
5	n	u(n)	v(n)	u(n)/ u(n-1)	v(n)/ v(n-1)				
6	0	100	100						
7	1	110	112	1.1	1.12				
8	2	121	125.44	1.1	1.12				
9	3	133.1	137.44	1.1	1.0957				
10	4	146.4	150.88	1.1	1.0978				

Figure 10.

Identifying x with 1+R, we see that a 1-month delay has reduced a 100% exponential growth rate to about 61.8%.

This illustrates the fact that the introduction of delay in a system undergoing exponential growth leads to a system that tends toward exponential growth—but at a reduced rate. Such calculations are readily pursued via spreadsheet (Figure 10). For example, "Should I accept my bank's offer to replace 10% annual interest by 12% based on my beginning balance of 2 years ago?"

Much as introducing delay will slow exponential growth, so does *eliminating* delay *increase* the rate of exponential growth. In modern economic theory, facilitating debt has emerged as a powerful way of eliminating delays built into traditional economic systems, thereby stimulating economic growth. For those who see continued economic growth as a way of accommodating future increases in world population, the manipulation of debt continues to be an important tool. In some sense, such economic principles go back to Fibonacci!

Matrix Methods

Another important idea that emerges from Fibonacci is the importance of breaking populations down into cohorts—e.g., babies and adults. One way of implementing this is to represent Fibonacci's rabbit problem in matrix notation. Letting b(n) and a(n) denote number of baby and adult pairs, respectively, the entire population can then be represented by a

2 x 1 matrix **u**(n) = $\begin{bmatrix} \mathrm{b(n)} \\ \mathrm{a(n)} \end{bmatrix}$.

The conditions of Fibonacci's problem

$$b(n) = a(n-1) \quad \text{and} \quad a(n) = b(n-1) + a(n-1)$$

can now be formulated in matrix form as

u(n) = **Tu**(n-1)

where **T** is the 2 x 2 matrix $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ and **u**(0) = $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

The fact that matrix multiplication is associative leads to the closed form solution

u(n) = $\mathbf{T}^{\mathrm{n}}$ **u**(0)

By way of generalization, suppose that adult pairs have a 20% mortality rate, baby pairs have a 10% mortality rate, and only half the adult pairs have a pair of babies each month. This situation can now be represented as

$$\begin{bmatrix} b(n) \\ a(n) \end{bmatrix} = \begin{bmatrix} 0 & .5 \\ .9 & .8 \end{bmatrix} \begin{bmatrix} b(n-1) \\ a(n-1) \end{bmatrix}$$

or

$$\begin{bmatrix} b(n) \\ a(n) \end{bmatrix} = \begin{bmatrix} 0 & .5 \\ .9 & .8 \end{bmatrix}^n \begin{bmatrix} b(0) \\ a(0) \end{bmatrix}$$

At this point it becomes desirable to bring computer technology to bear on modeling the (non-whole number) growth of **u**(0) (see Figure 11).

Applying these ideas to a human population with perhaps 20 age-based cohorts leads to an important demographic tool known as the Leslie matrix.

With such matrix tools at hand, it becomes important to develop population models that take economic well-being into account. For if the Malthusian specter of "checks of vice and misery" were to be realized, they would be felt by the world's poor long before long before reaching those near the top of the economic ladder. This

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P
1	**Letting u(n) = b(n) + a(n), this spreadsheet uses matrix multiplication to allow for fertility and mortality rates that determine the growth of Fibonacci's generalized rabbits.**															
2																
3	adult fertility =		0.5													
4	adult mortality =		0.2													
5	infant mortality =		0.1													
6			n=	0	1	2	3	4	5	6	7	8	9	10	11	12
7		0	0.5	1	0	0.45	0.36	0.49	0.55	0.66	0.78	0.92	1.09	1.29	1.52	1.8
8		0.9	0.8	0	0.9	0.72	0.98	1.11	1.33	1.56	1.85	2.18	2.58	3.04	3.59	4.24
9				u(n)=	0.9	1.17	1.34	1.6	1.88	2.23	2.63	3.1	3.67	4.33	5.11	6.04
10			u(n)/u(n-1)=			1.3	1.15	1.19	1.18	1.18	1.18	1.18	1.18	1.18	1.18	1.18

Figure 11.

suggests dividing the world's population into economic cohorts (e.g., seven cohorts of 1 billion each) and calculating their interaction as well as the population dynamics of each cohort.

Migration

An important mechanism for interaction between economic cohorts is human migration. People move from where it is worse to where it is better, much like energy flows from where it is hotter to where it is colder (to do otherwise would violate the second law of thermodynamics). In order to pursue such an analogy in mathematical terms, we will need a numerical "measure of poorness" that corresponds to temperature.

In an agrarian world, "number of people per acre of arable land" might be such a measure of poorness. But in a world where access to fossil fuels, fertilizer, and technology modifies the importance of arable land, "people per megawatt of available energy" (including solar energy) may provide a more appropriate measure.

An instructive way to pursue this analogy is to model "heat flow in a rod." Dividing a rod into seven segments and assigning an initial temperature to each, we could let u(t,n) denote the temperature of the n-th segment at times t = 1, 2, 3,…. Then, in keeping with Newton's law of cooling, heat flow will be determined by the *differences* in temperature between each segment and its adjoining neighbors. That is,

$$u(t+1, n) - u(t, n) = r[u(t, n) - u(t, n-1)] - r[u(t, n) - u(t, n+1)]$$

where r (0 < r < .5) denotes the conductivity of the rod. Writing this equation as

(2) $u(t+1, n) - u(t, n) = r[u(t, n-1) - 2u(t, n) + u(t, n+1)]$

we recognize (2) as the diffusion equation

$$\frac{\partial u}{\partial t} = r\frac{\partial^2 u}{\partial x^2}$$

in discrete form. It is readily modeled on a spreadsheet, using fictional elements in columns B and J to impose boundary conditions in Figure 12.

In a Lineland world one can posit an analogous process, with people moving to adjoining countries in proportion to differences in well-being on the two sides of the border. But in our spherical world people are not restricted to migrating to a neighboring country. A first step toward accommodating this fact is to write the diffusion rule (2) in matrix form.

$$\begin{bmatrix} u(t+1,1) \\ u(t+1,2) \\ u(t+1,3) \\ u(t+1,4) \\ u(t+1,5) \\ u(t+1,6) \\ u(t+1,7) \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 & 0 & 0 & 0 & r \\ r & 1-2r & r & 0 & 0 & 0 & 0 \\ 0 & r & 1-2r & r & 0 & 0 & 0 \\ 0 & 0 & r & 1-2r & r & 0 & 0 \\ 0 & 0 & 0 & r & 1-2r & r & 0 \\ 0 & 0 & 0 & 0 & r & 1-2r & r \\ r & 0 & 0 & 0 & 0 & r & 1-2r \end{bmatrix} \begin{bmatrix} u(t,1) \\ u(t,2) \\ u(t,3) \\ u(t,4) \\ u(t,5) \\ u(t,6) \\ u(t,7) \end{bmatrix}$$

Here the diagonal element in the j-th row and j-th column represents fraction of the population in country j that remains in country j. The element in row j and column k represents the fraction of people in country k that migrate to country j. Such an interpretation lends itself to allowing non-negative numbers for *all* matrix entries. A nonnegative entry a_{jk} in the j-th row and k-th column denotes the fraction of people in country k that migrates to country j. Neglecting births and deaths, "conservation of people" originating in country k requires that the elements in column k sum to 1. In this case, this model for migration is a Markov chain of the form **u**(t+1) = **Tu**(t) where **T** is a *stochastic matrix.*

Logistic Growth

A homogeneous population that enjoys unlimited space and resources may be well modeled by the exponential growth difference equation

$$u(n) - u(n-1) = Ru(n-1)$$

However, this is not the case with a population whose environment has a finite carrying capacity, and here ecologists are likely to turn to a modification of (1) called *logistic growth*. As with exponential growth, logistic growth can be put into a financial context.

Consider a bank (let's call it Murky Savings Bank) that offers a generous annual interest rate R but also imposes "a small service fee" that is determined by a constant E (e.g., R = .1 and E = .005). As in (1), the interest payment at the end of the n-th year is to be proportional to the beginning balance u(n-1). However, the service fee is to be proportional to the *square* of u(n-1). At such a bank the growth of an initial deposit u(0) would be determined by the difference equation[3]

$$(3) \quad u(n) - u(n-1) = Ru(n-1) - Eu(n-1)^2$$

Recalling that the exponential growth equation has a closed form solution $u(n) = (1 + R)^n u(0)$ it is natural to ask whether there is an analogous formula for solutions of (3). Unhappily no such closed form solution exists. Instead, solving (2) requires repeated application of the recursive rule

$$(4) \quad u(n) = (1 + R)u(n-1) - Eu(n-1)^2$$

Turning to a spreadsheet to implement (4) for R = .1, E = .005, and u(0) = 5, we find that the values of the u(n) correspond to the sigmoid (s-shaped) graph given in Figure 13.

An important insight into this solution of (3) follows from the question, "Can the fee ever overtake the interest?" From the inequality

$$Eu^2 > Ru$$

we see that if u > R/E then the fee exceeds the interest. If 0 < u(0) < R/E, then the solution of (4) will approach, but never reach R/E. Exceptions to this typical behavior arise in "chaos theory."

In light of the central role played by R/E, it be comes useful to define a "target" T = R/E and to reformulate the difference equation (3) accordingly. Writing (3) as

$$u(n) - u(n-1) = E[R/E - u(n-1)]u(n-1)$$

	A	B	C	D	E	F	G	H	I	J	K
1	This spreadsheet models heat flow in a rod whose tempreature is measured at seven discrete points. It implements the diffusion rule u(t,n) = u(t-1,n) + r[u(t-1,n-1) - 2u(t-1,n) + u(t-1,n+1)]										
2											
3	r=	0.1									
4											
5	t\n		1	2	3	4	5	6	7		
6	0	270	20	280	170	120	120	30	270	20	
7	1	221	71	243	176	125	111	63	221	221	
8	2	205	103	219	178	129	108	83.6	205	205.2	
9	3	193	125	203	177	131	107	98.2	193	193	
10	4	184	140	193	175	134	109	109	184	183.6	
11	5	176	149	186	173	135	111	116	176	176.1	

Figure 12.

	A	B	C	D	E	F	G	H
1	This spreadsheet allows you to assign values to R, E, and u(0). It models the logistic growth equation u(n) - u(n-1) = Ru(n-1) - Eu(n-1)².							
2								
3	n	u(n)	R=	0.1	E=	0.005	R/E=	20
4	0	5						
5	1	5.375						
6	2	5.768047						
7	3	6.1785						
8	4	6.60548						
9	5	7.047867						
10	6	7.504291						
11	7	7.973148						
12	8	8.452608						
13	9	8.940636						
14	10	9.435024						
15	11	9.933428						
16	12	10.43341						
17	13	10.93247						
18	14	11.42812						
19	15	11.91792						
20	16	12.39953						
21	17	12.87074						
22	18	13.32954						
23	19	13.77411						
24	20	14.20289						
25	21	14.61457						
26	22	15.00809						

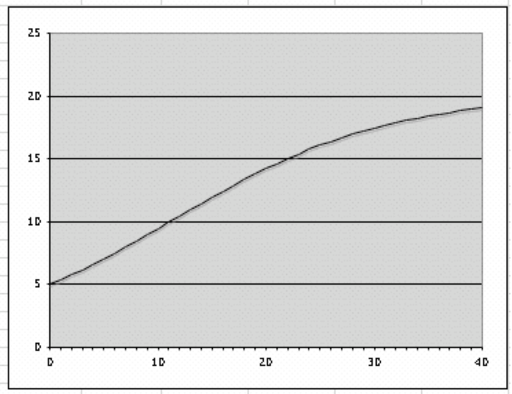

Figure 13.

we arrive at

$$(2') \quad u(n) - u(n-1) = E[T - u(n-1)]u(n-1)$$

as an equivalent rule for logistic growth. As the u(n) approach T, the term in brackets approaches zero, as do the changes u(n) - u(n-1) (Figure 14).

As with exponential growth, it is of interest to inquire into the effect of delays on solutions of (2). Here we have seen that solutions of the delay exponential equation

$$(5) \quad u(n) - u(n-1) = Ru(n-1-d)$$

approach exponential growth, but with a growth rate smaller than R. So what would be the effect of injecting a delay into the logistic difference equation (3) whose solutions exhibit sigmoid behavior?

Because there are two terms on the right side of (4), there are several ways of injecting delay. For example, we can inject delays into the first term, leading to

$$(6) \quad u(n) - u(n-1) = Ru(n-1-d) - Eu(n-1)^2$$

[3] In an ecological setting, the expression $- Eu(n-1)^2$ can be thought of as a form of "environmental damping" that is being imposed on growth that would otherwise be exponential.

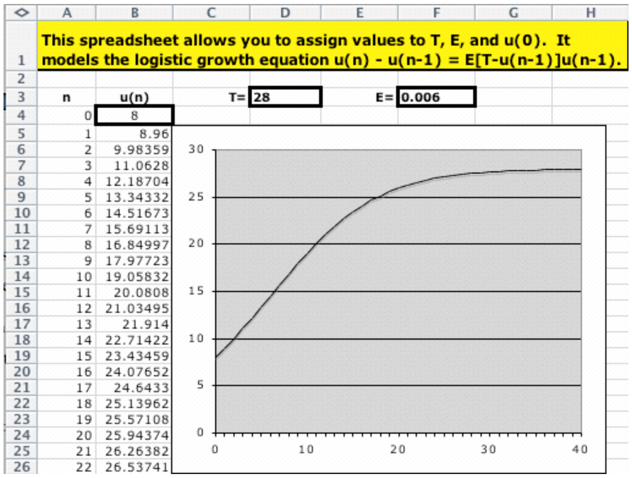

Figure 14.

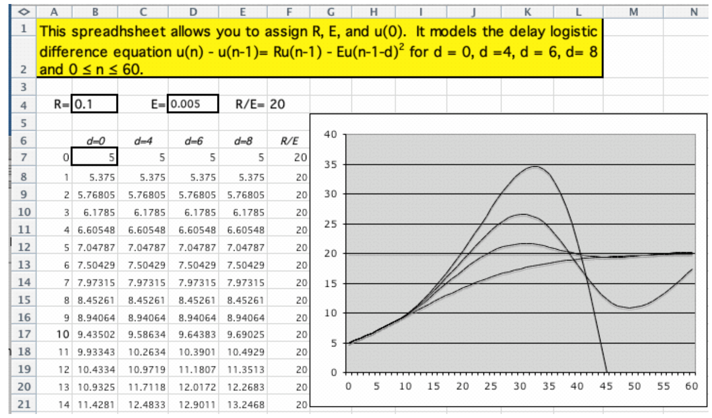

Figure 15.

As with (5), such delays can be expected to slow the growth of the u(n).

But what about injecting delay into the damping term, as in

(7) $u(n) - u(n\text{-}1) = Ru(n\text{-}1) - Eu(n\text{-}1\text{-}d)^2$?

In ecological terms, there is a provocative context for (7). While (3) may be a realistic model for the growth of yeast in a closed jar, humans are much cleverer than yeast. Even though we are constrained by a finite earth, our industrial civilization has been able to inject delays into the environmental damping term that imposes the limit R/E on lesser species.

Using a spreadsheet (Figure 15) to model (7) we find that $d > 0$ does indeed enable us to exceed the carrying capacity R/E—but with a price to be paid down the line.

While models such as (3) and (7) should be taken with a grain of salt, they can help us enlarge the vocabulary that we use in discussing global change. Malthus and (1) gave us *exponential growth* while Verhulst and (3) gave us *logistic growth* (with a "soft landing" at R/E). What (7) injects into the discussion are the phenomena *overshoot and oscillation toward equilibrium* and *overshoot followed by crash*.

A Copernican Metaphor

Having shifted the discussion from celestial change to terrestrial change, it seems appropriate to raise the question of connections between the two. To be sure, numeracy and "the mathematics of change" are common to both. But are there other connections of a more human nature?

In pursuit of this question we viewed Episode 6 from the BBC series on "The Ascent of Man." Here, under the heading "The Starry Messenger," Jacob Bronowski recounts events that led to abandonment of the geocentric framework and how Galileo's efforts to promulgate the Copernican system led to his trial before the inquisition. While filled with social turmoil, these events also set the stage for scientific and industrial revolutions that have shaped our world. Might there be comparable resistance to accepting the implications of terrestrial numeracy?

One way of pursuing this question is to consider "a Copernican metaphor" in which students are confronted with the following framework.

Category	**Celestial World**	**Terrestrial World**
synodic view	geocentrism	______________
sidereal view	heliocentrism	______________
great book	*De Revolutionibus…*	______________
technology	telescope	computers
orthodoxy	religion	______________
martyrs	Giordano Bruno	______________
heroes	______________	______________
___________	______________	______________
___________	______________	______________
___________	______________	______________

As candidates for a synodic view of the terrestrial world we considered both *plutocentrism* (viewing the world from an island of great wealh) and *anthropocentrism* (building on the biblical idea that man shall have dominion "over all the earth and over every creeping thing that creeps on the earth"). This led to questions about the orthodoxies and great books that support such views.

But in trying to stimulate such discussion, it is also important to focus on ways in which the metaphor breaks down. Here we noted that Kepler's laws deal with a deterministic system while terrestrial change is believed to involve free will. Indeed, it is the belief that there are important decisions to be made that justify our efforts to develop the mathematics of global change.

Call for Papers

The "theme" of the fall issue of the *Journal of Mathematics Education at Teachers College* will be *Evaluation*. This "call for papers" is an invitation to mathematics education professionals, especially Teachers College students, alumni and friends, to submit articles of approximately 2500-3000 words describing research, experiments, projects, innovations, or practices related to evaluation in mathematics education. Articles should be submitted to Ms. Krystle Hecker at JMETC@tc.columbia.edu by January 21, 2012. The spring issue's guest editor, Ms. Heather Gould, will send contributed articles to editorial panels for "blind review." Reviews will be completed by February 1, 2012, and final drafts of selected papers are to be submitted by March 1, 2012. Publication is expected by April 15, 2012.

Call for Volunteers

This *Call for Volunteers* is an invitation to mathematics educators with experience in reading/writing professional papers to join the editorial/review panels for the spring 2012 and subsequent issues of *JMETC*. Reviewers are expected to complete assigned reviews no later than 3 weeks from receipt of the manuscripts in order to expedite the publication process. Reviewers are responsible for editorial suggestions, fact and citations review, and identification of similar works that may be helpful to contributors whose submissions seem appropriate for publication. Neither authors' nor reviewers' names and affiliations will be shared; however, editors'/reviewers' comments may be sent to contributors of manuscripts to guide further submissions without identifying the editor/reviewer.

If you wish to be considered for review assignments, please request a *Reviewer Information Form*. Return the completed form to Ms. Krystle Hecker at hecker@tc.edu or Teachers College Columbia University, 525 W 120th St., Box 210, New York, NY 10027.

Looking Ahead

Anticipated themes for future issues are:

Spring 2012	Evaluation
Fall 2012	Equity
Spring 2013	Leadership
Fall 2013	Modeling
Spring 2014	Teaching Aids

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