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Forward-Thinking Orientations for Mathematics Education

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NOTES FROM THE FIELD

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Multiplication by Sunlight: How Can a Geometric Definition be Realized in a Physical Tool?

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ABSTRACT Physical models for exploring multiplication are fixtures in elementary classrooms. The most widely used physical models of multiplication are collections of discrete things, such as Cuisenaire rods, Unifix Cubes, or Base-10 Blocks. But discrete physical models are limited in the products they can represent. By contrast, pictorial models, such as number lines or area models, are continuous and thus represent a broader range of products. However, pictorial models are limited in how they can be manipulated. The discrete/continuous divide across physical/pictorial representations of multiplication frames the overarching design problem that motivated our work: How could a physical, manipulable tool realize a continuous model of multiplication? This is a significant problem because, to our knowledge, there are no examples of physical models of multiplication that offer the plasticity of pictorial models. We describe one such model here—an analog technology that we refer to as a *Sunrule*. We explain the design of the device and report an initial instructional activity where pre-service teachers explored the device in groups.

KEYWORDS *multiplication, sunlight, physical models, sun shadows*

Introduction

The gradual accumulation of knowledge about multiplication in school follows a known trajectory. Basic facts, such as times tables, are memorized. From there, students learn a collection of algorithms for calculating the products of different types of numbers, including multidigit integers, decimals, and fractions. One takeaway for students is that multiplication is a collection of rules that apply in different circumstances. Indeed, elementary students and teachers alike tend to have procedural dispositions toward multiplication (Fuson, 2003; Hiebert, 2013; Lampert, 1986).

To help children develop a deeper familiarity with multiplication, some teachers rely on physical or pictorial models (Kosko, 2019). By *physical models* of multiplication, we mean tangible, graspable manipulatives such as Unifix cubes, Cuisenaire rods, and Base-10 Blocks. Such models are fixtures in elementary classrooms because it is believed that "children need opportunities to work with objects in the physical world before they will be ready to work with pictures and other representations" (Reys et al., 2014, p. 25). Physical models are generally *discrete* because they are collections of fixed quantities (e.g., a set number of cubes or blocks, a set of rods of specific heights; Kosko, 2019). By contrast, *pictorial models,* such as area or number line representations, are generally *continuous* because they are not limited to a fixed set of pre-determined things. Elementary teachers generally use discrete models to represent multiplication (Kosko, 2019).

While physical models offer tangible representations of multiplication, discrete objects are limited in the types of products they can represent. Meanwhile, continuous pictorial models can represent a set of unlimited products; however, such models cannot be investigated tangibly to the same extent as their physical counterparts. The discrete/continuous divide across physical/pictorial representations of multiplication frames the overarching design problem that motivated our work: How could a physical, manipulable tool realize a continuous model of multiplication? This is a significant problem because, to our knowledge, there are no examples of physical models of multiplication that offer the plasticity of pictorial models. We sought to explore how a model of multiplication that combined the tangibility of a physical thing with the flexibility of a picture might create new opportunities for exploring multiplication.

Design of the Sunrule

To model continuous multiplication using a physical object requires some method for increasing and decreasing lengths. We refer to this as the *variable length* design problem. The historical solution to this problem was the *slide rule,* an arithmetic aid that reigned from the 17th century until it was abandoned for electronic calculators in the 1970s (Tympas, 2017). Slide rules are ingenious, powerful devices that deserve a place in mathematics classrooms. However, as their logic of use is circumscribed by the theory of logarithms[1], they are not suitable for helping elementary-age students explore multiplication[2]. How else could adjustable lengths be used to model multiplication with a physical tool? Our answer to this question was based on McLoughlin and Droujkova's (2013) diagrammatic definition that frames multiplication as *continuous directed scaling*—i.e., the length of one segment is a positive or negative multiplier that scales the length of another segment in the positive or negative direction. Their diagrammatic definition of multiplication was inspired by Hilbert's treatment of segment multiplication (Hilbert, 1999). Dimmel and Pandiscio (2020) illustrate how the product of two segments can be constructed with a compass and straightedge.

To design the *Sunrule*, we used sunlight as a straightedge. According to Decamp and Hosson (2012), sunlight offers a readily available, renewable, and abundant supply of naturally occurring parallel rays. We recognize that the sun's rays are not entirely parallel. Still, over small distances, sunlight is several orders of magnitude more parallel than other real-world examples of parallel lines, such as railroad tracks. The sun's parallel rays mean that the height of any shadow-casting object is proportional to the length of its shadow, and, for a given altitude of the sun, this proportion is the same for all object-shadow pairs (Douek, 1999). Thus, multiplying numbers, in general, requires control over the position of the sun. We refer to this as the *variable altitude* design problem.

While the sun cannot be moved, there is a solution to the *variable altitude* problem: We can change the apparent altitude of the sun by varying the angle of inclination of a surface onto which shadows are cast. By increasing the angle of inclination of a surface, we decrease the lengths of any shadows falling upon it; by decreasing the angle of inclination, we would increase the lengths of those shadows. Thus, by varying angles of inclination, it is possible to control the apparent altitude of the sun from 90 degrees (i.e., directly overhead, no shadow) to 0 degrees (i.e., sun on the horizon, undefined shadow). Below, we explain how the *Sunrule* can be used for multiplication and illustrate how it solves the *variable length* and *variable altitude* design problems.

Multiplication by Sunlight

The *Sunrule* models multiplication by casting shadows. It is not a combination of a sundial and slide rule; however, the name is apt because it combines essential elements of each tool (e.g., gnomons[3], adjustable scales) in novel ways. A *Sunrule* consists of two ruled gnomons affixed at right angles to a ruled surface—i.e., the *shadow plane.* The device is independent of any particular choice of unit (e.g., inches, cm, mm), in the sense that the unit could be of any height, and that unit determines the other rulings. It is sufficient for the unit gnomon and the adjustable gnomon to be ruled in equal increments; the surface could have other (equally spaced) increments as rulings. We have found it convenient to use the same unit increment for the gnomons and the shadow plane, though we recognize that this is a design parameter that could be varied and explored.

For the *Sunrule* shown in Figure 1, there is a longer gnomon on the bottom and a shorter gnomon on the top. The shorter gnomon functions as a unit length. The unit length and the factor by which its shadow was scaled define a multiplier; in this case, that multiplier is 3. The

1 A slide rule's sliding scales are ruled in logarithmic increments, which allow products (multiplication) and quotients (division) to be expressed as sums (addition) and differences (subtraction).

2 This is a comment on the elementary mathematics curriculum, not a statement about the capacity for children to learn the theory of logarithms.

3 This is the name for the part of a sundial that casts a shadow.

Figure 1

A Sunrule *constructed by elementary teacher candidates*

PHOTO BY MEG PANDISCIO (2020)

device is positioned so that the length of the unit shadow extends to 3 units. The height of the longer gnomon can be adjusted by sliding it up or down; the height of this gnomon specifies the multiplicand, which is 4, in this example. The product, 12, is given by the length of the shadow of the adjustable gnomon. A video demonstration of how to construct a *Sunrule* is available at: https://tinyurl.com/wnjky5e4

The *variable length* and *variable altitude* problems solved by the *Sunrule's* design prescribe two movements that can be used to transform multiplication problems. One movement is that the angle of inclination to the sun can be varied by tilting the device up or down. Note that the gnomons in Figure 2 are in the same positions as those shown in Figure 1. That is, we left the multiplicand gnomon at four units. But we tilted the device so that the length of the shadow of the unit gnomon was two units. With that, the angle of inclination of the device toward the sun increases. This increase in inclination decreased the length of the unit gnomon's shadow from 3 units to 2 units; therefore, the multiplier changes from 3 to 2. As a result, Figure 2 shows 2 × 4 = 8.

The *Sunrule* provides a material context that models multiplication as a scaling operation. Because the tilt of the device can be varied with continuous movements, the device multiplies fractions as readily as integers. This is a potentially significant affordance because fractions

Figure 2

A Sunrule *that shows 2 × 4 = 8*

PHOTO BY MEG PANDISCIO (2020)

Figure 3

A Sunrule *that shows 1.5 × 3 = 4.5*

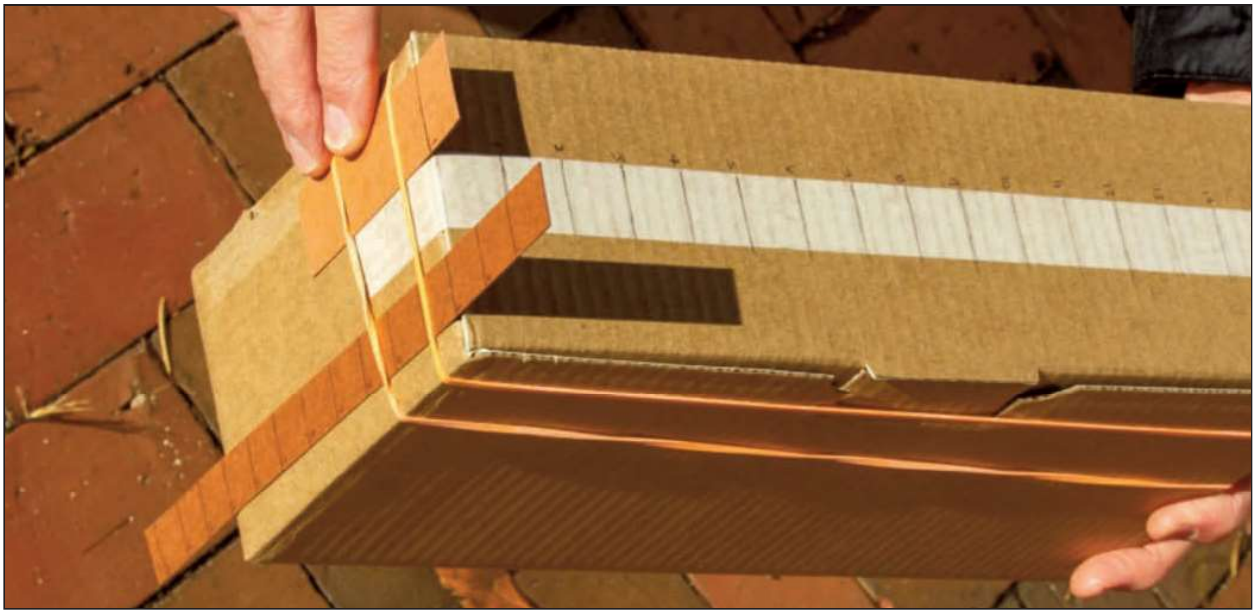


PHOTO BY MEG PANDISCIO (2020)

Note. In this image, a *Sunrule* has been inclined so that the length of the unit shadow is 1.5 units, the height of the multiplicand gnomon is three units, and the adjustable gnomon's shadow is 4.5 units; hence, 1.5 × 3 = 4.5.

are an endless source of difficulty for students (Sidney et al., 2019). Figure 3 shows 1.5 × 3 = 4.5. This is just one example of a fractional product defined between 3 and 4; other products could be modeled by slightly changing the angle of inclination. The multiplier would be slightly less than or slightly more than 1.5, and the length of the multiplicand shadow would vary proportionally. It is not any particular fractional product but rather the ability to move between fractional products—in an arithmetic analogy of the continuous transformations that can be used to explore dynamic geometry diagrams—that is potentially significant.

We do not propose that the *Sunrule* should replace other models for teaching multiplication; however, because it is a physical model that allows for products to be explored through continuous variation—by (a) varying the height of the adjustable gnomon and (b) varying the angle of inclination of the shadow plane—it warrants investigation. In an initial effort to gauge its pedagogical value, we facilitated an activity with elementary teacher candidates.

Initial Teaching Activity with Elementary Mathematics Teacher Candidates

During Fall 2020, the second author taught two sections of an elementary mathematics methods course, each with five students enrolled, that met on different days. The *Sunrule* investigation was planned as a two-lesson activity. For the first part of the activity, students worked with the second author to build *Sunrules*. Students were told that the device was for a mathematics exploration and that it needed to be used outside on a sunny day. Students were not told that the *Sunrule* was designed to model multiplication because we were interested in how students would explore and make sense of the device. Both sections of the course completed the first part of the activity; however, one group did not complete the second part because of inclement weather.

For the second part of the activity, students explored their *Sunrules* outside in two small groups. For safety, they wore masks and followed social distancing protocols. We used fixed video cameras to record the activity of each group. The second author moved back and forth between the groups to facilitate their explorations of the device. Using a semi-structured protocol, he provided directed guidance to the groups of students. An example of a directed question was: What are the ways that the lengths of the shadows of the gnomons could be varied? The purpose of this question was to help students identify the two movements through which the shadows could be varied to model products. The second author posed questions from the protocol to each group, as needed, to keep the students from getting stuck and guided them toward investigations of its mathematical affordances (i.e., a physical model that displays multiplication as a scaling operation). In the following episodes, we discuss how students explored and interacted with the *Sunrule*.

Episode 1: Sara's initial encounter with the *Sunrule*

One group consisted of two students, Zak and Sara[4]. The second author launched the exploration activity for them by asking, "Any idea what this box does?" Although Sara declared that she did not know what the device did, she oriented the device in the intended way. Figure 4 shows how she positioned the *Sunrule* so that it was aligned with the azimuth of the sun (i.e., the compass heading of the sun, or the place where the sun would appear if it were brought down to the horizon).

This caused the shadows of the gnomons to fall parallel to the strip of rulings on its surface. In that instance, Sara may not have understood the mathematical affordances of the device, but she instinctively positioned it correctly. Then, she changed the device's angle of inclination by tilting the device toward and then away from the sun. This caused the shadows of the gnomons to shorten and then lengthen, respectively, as shown in Figure 5.

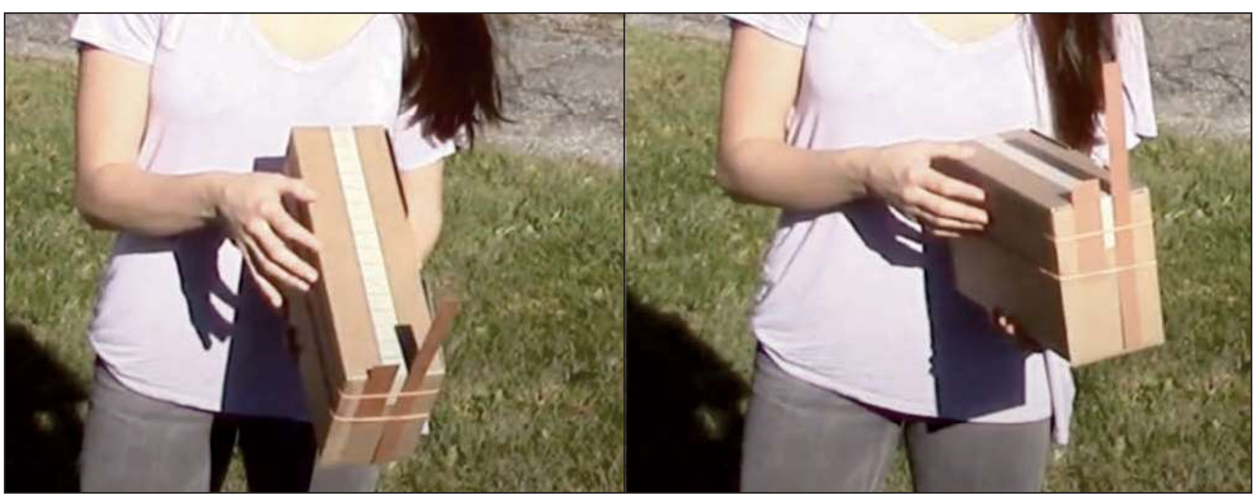
As Sara varied the angle of inclination, she and Zak speculated that the device indicated a relationship between the sun and the shadows. Sara noted the significance of the angle of inclination to the length of the shadows. She said, "It really depends on how you hold it, like, if you tilt it towards [sic] the sun, then the shadows become very short. If you tilt it away from the sun, the shadows get a lot longer." These initial interactions that varied the lengths of the shadows by changing the angle of inclination are the core of the mathematical design of the *Sunrule*. This feature was salient for Sara almost immediately. Her recognition of the significance of tilting the device suggested that the grounding predicate for the geometric definition of multiplication is a natural and potentially powerful embodiment for a continuous scaling conception of multiplication.

Figure 4

Sara orienting the Sunrule *so that the shadows of the gnomons are aligned with the rulings on the shadow plane*

Figure 5

Sara increases the angle of inclination (left frame) and then decreases the angle of inclination (right frame), causing the lengths of the shadows to decrease (left frame) and increase (right frame)



[4] All names are pseudonyms.

Episode 2: Varying the angle of inclination

The second group of students consisted of Sheila, Martha, and Donna. Like Sara, Sheila explored the device by varying its angle of inclination to the sun. After Sheila, Martha, and Donna explored the device for a few minutes, Sheila shared her observation with the group. She said, "When the big one is at 4, and the little one is at 1, so then when you make the big one 8 and the little one 2…they change by the same amount respective to each other." As she said this, Sheila adjusted the angle of the inclination of the device away from the sun. This caused the shadows of the gnomons to lengthen in a movement that was similar to how Sara changed the tilt of the device.

The second author asked the group for the name of the operation demonstrated by varying the lengths of the shadows. Martha replied, "I don't think we know what it is called." Sheila, like Sara, focused on the variability of the angle of inclination as a key affordance of the device. Sheila linked this general observation to specific pairs of numbers (1, 4) and (2, 8) but could not identify the mathematical relationship she had noticed.

After 10 minutes of exploration, both groups had observed that the angle of inclination of the device determines the ratio between the height of a gnomon and the length of its shadow. For example, Sara observed that if the longer gnomon were set to be two or three times the height of the shorter gnomon, that difference in height would be carried through the variations in the lengths of the shadows as the angle of the inclination was changed. The second author assembled the groups in a socially-distanced semicircle because neither group had connected their observations about ratio to multiplication. He summarized the ratio ideas each group had discussed and told them that the device models multiplication. His decision was framed by the reality that this investigation occurred within the context of an elementary methods class. He wanted to ensure that the teacher candidates would recognize the *Sunrule* as a physical, tangible model of multiplication. In future studies with elementary teacher candidates, we would allow more time for open-ended exploration of the device.

Episode 3: Modeling division with the *Sunrule*

In their discussion of multiplication, Zak and Sara realized that the device could also be used to represent division. Zak demonstrated this idea by showing how the multiplication problem 2 × 5 = 10 could be interpreted as the division problem 10 ÷ 5 = 2. To use the *Sunrule* to divide two numbers, let the height of the multiplicand gnomon be the divisor and then vary the angle of inclination, so the length of its shadow is equal to the dividend. The quotient will then be given by the length of the unit gnomon's shadow. Zak demonstrated this while narrating his *Sunrule* manipulations. Sara and Zak's explorations of the connection between multiplication and division underscore the rich pedagogical opportunities of the *Sunrule*.

Discussion

Conceptual explorations of multiplication

A feature of the *Sunrule* is that its use of movement creates opportunities to differentiate *multiplicands* from *multipliers.* The angle of inclination of the shadow plane determines the multiplier, and the height of the adjustable gnomon determines the multiplicand. Thus, the *Sunrule* creates the possibility of problematizing commutativity because multiplying 2 × 3, for example, involves different movements than multiplying 3 × 2. In the former case, the shadow plane is inclined so that the length of the unit shadow is two and the height of the multiplicand gnomon is 3; whereas in the latter case, these movements are reversed. That the product in each instance is six may not be surprising for students, especially if they have had experience with multiplication. But the physical differences between multiplier and multiplicand could suggest questions for discussion: *Why can changes in the angle of inclination of the shadow plane be offset by adjusting the height of the gnomon? Does this work for all products? Why or why not?*

A second activity could help students explore families of products. Questions such as *What are the ways to make 15 by multiplying two numbers?* are fixtures in elementary mathematics classrooms. With the *Sunrule*, such questions could be explored in new ways. In particular, because the *Sunrule* is a continuous model of multiplication, it could lead students to consider not only pairs of whole number factors whose product is a given number but also pairs of rational numbers. For example, from 3 × 5 = 15, one could slightly decrease the angle of inclination of the shadow plane to increase the multiplier and then slightly decrease the height of the multiplicand gnomon in such a way that its shadow is still 15 units long. This could lead to pairs of rational factors of 15, like $3\frac{1}{3}$ and $4\frac{1}{2}$, or $3\frac{3}{5}$ and $4\frac{1}{6}$. A challenge would be reading the fractions from the incremental rulings. Still, even if their exact values were difficult to read from the rulings, the existence of such rational factors would be evident from the interplay of the shadows. This could lead to discussions of how many pairs of rational numbers there are whose product is a given number. Such activity could help blur the boundaries between rational and whole numbers (Dimmel & Pandiscio, 2020) and provide an opportunity for students to explore how families of products can be related by continuous variation.

Connections to geometry

The *Sunrule* is like a dynamic diagram in that it allows for the exploration of families of products through continuous movements. Equally significant, the device is a rare example of a mathematical tool optimized for use outdoors. The *Sunrule* harnesses the unique geometry of sunlight to create parallel shadows, the lengths of which are controlled by varying the angle of inclination to the sun to set the multiplier and by adjusting the height of the longer gnomon to set the multiplicand. This report focused on arithmetic descriptions of multiplication, but the *Sunrule* also creates opportunities for exploring multiplication geometrically. Such explorations would be appropriate for secondary students, for whom the *Sunrule* could create opportunities to examine how multiplication, similar triangles, and proportionality are related. For example, one activity for secondary geometry students would be to explore the multiplicative identity. Questions such as, *At what apparent altitude(s) of the sun will the multiplier be 1?* and, *What is the relationship between the apparent altitude of the sun and the magnitude of the multiplier?* could create opportunities for geometry students to probe the trigonometric applications of the device. The *Sunrule* also creates an opportunity for teachers of geometry to celebrate sunlight as the quintessential real-world example of parallel lines. Another series of activities could be grouped as design questions. Examples in this category might be, *Do the gnomons need to be perpendicular to the shadow plane? If the gnomons do not need to be perpendicular, what are the requirements for the position of the gnomons? and, What happens if we change the unit length?*

Limitations

The principal limitations of the *Sunrule* concern its accuracy. We have identified four inherent physical defects that introduce errors in its calculation. By inherent, we mean these defects are a consequence of their physicality—they can be managed but never eliminated. The first source of error is the angle that the gnomons make with the shadow plane. The closer the gnomons are to perpendicular, the greater the accuracy. The second source of error is the flatness or uniformity of the shadow plane. The closer this is to perfectly flat, the greater the accuracy. The third source of error is the resolution, or sharpness, of the shadows. The fourth limitation concerns the accuracy with which the gnomons and the shadow plane are ruled and marked. This constellation of physical errors leads to another limitation: The *Sunrule* can only effectively model a relatively small range of products. For example, the *Sunrules* described in this report had a maximum length of 20 units, which meant that any product greater than 20 would be *off the board.* A possible solution to the limited range problem is to use place value, so *18 × 20* would be off the board, but *1.8 × 2* would not. This is how slide rules were able to multiply a wide range of numbers on relatively small scales. But the reliability of these calculations depends on minimizing the inherent errors.

Although the inherent material flaws affect the accuracy of the multiplication, the overall process of multiplying and the relationship between the multiplier, multiplicand, and product can be explored with the device. The gnomons and shadow planes are easily manipulated to display a range of numerical combinations. We are exploring various designs and production quality choices that would minimize the errors and maximize the range of numbers that can be multiplied. We envision a version where the markings are as precise as a standard school ruler.

Conclusion

The *Sunrule* uses sunlight, an affordance of the world, to model multiplication, a mathematical operation. Simultaneously, it shares a mathematically valid and robust representation of multiplication that is often missing in elementary school classrooms—multiplication as continuous scaling (Dimmel & Pandiscio, 2020; Kosko, 2019). By using a feature of the world to build a mathematical model, the *Sunrule* represents an inversion of what is typically encountered in real-world mathematics.

The COVID-19 pandemic has reconfigured social life. For schools, this has meant adapting instruction to remote, hybrid, or outdoor modalities, among other innovations, some of which may endure even when COVID-19 has been mitigated. The *Sunrule* provides a concrete material context for doing a mathematical activity outside—not simply for the sake of being outside, but because being outside is essential to use the device to do mathematical work. It is a variable, tangible device for modeling families of multiplication problems and probing their mathematical structure. Beyond arithmetical utility, activities with the *Sunrule* could pull students away from screens and create opportunities for students and teachers to reflect on how the geometry of sunlight is integrated with its design. These would be desirable outcomes at any time, and they are especially urgent in the face of the disruptions to teaching and learning brought on by the pandemic.

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