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A CENTURY OF LEADERSHIP IN
MATHEMATICS AND ITS TEACHING

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Toward A Coherent Treatment of Negative Numbers

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The transition from whole numbers to integers involves challenges for both students and teachers. Leadership in mathematics education calls for an ability to translate depth of understanding into effective teaching methods, and this landscape includes alternative treatments of familiar topics. Noting the multiple meanings associated with the horizontal bar that is often referred to as "minus sign," the authors introduce a novel notation intended to address this ambiguity. In this system, the symbol "–" is reserved exclusively for subtraction. The four arithmetic operations and the concept of a number's opposite are then illustrated in light of such a notational shift. A valuable aspect of this excursion is that it encourages teachers to reflect on the relationship between some important mathematics and the pedagogical approaches they now use.

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Leadership in mathematics education calls for both mastery of the subject and an ability to translate depth of understanding into effective instruction. These skills should enable a leader to discern particularly informative approaches to the solution of a problem, the proof of a conjecture, and similar activities of fundamental importance. By way of example, leaders in mathematics education are likely to express a preference for "divide and average" (Kreith & Chakerian, 1999, pp. 1–55) as a method for approximating square roots. Not only is this more efficient than the opaque algorithms that are sometimes taught, it also constitutes a pre-calculus introduction to Newton's method, one that can be extended to the calculation of n-th roots and the solution of polynomial equations.

Leaders can also bring their skills to bear in the development of alternative treatments of common curricular topics, such as negative integers. The transition from whole numbers to integers can be a challenging one for both student and teacher. The difficulties involved in accommodating "negative numbers" are illustrated by the problem

Express $-(-3-5)$ as an integer.

While there exist a variety of techniques for helping students arrive at the answer 8, rarely is it acknowledged that the problem being posed uses the "minus sign" in three different ways. Working left to right, the horizontal bar preceding the parenthetical expression calls for taking the *opposite* of that expression; the bar preceding the symbol 3 is part of our representation of the *integer* "negative three"; the bar preceding the 5 calls for *subtraction* of the number 5 from negative three.

Providing leadership is this situation can also be a challenging task. Colleagues seeking help are likely to be looking for practical tools that can help their students solve textbook problems. Even with such tools at hand, teachers should be prepared to deal with conceptions of negative numbers that students may have developed on their own. And while the Common Core Standards emphasize use of the number line to help students and teachers navigate these stormy seas, they contain little guidance on how to achieve consistency and coherence in problems such as the one posed above.

In this situation, a willingness to go to the mathematical roots of the problem may be a crucial aspect of leadership. For while a formal account of the transition from whole numbers to integers is not part of the K–12 curriculum, an ability to relate pedagogical tools to their theoretical roots can be enlightening.

Negative Numbers Revisited

The addition and multiplication of whole numbers allow for simple and mathematically sound representations. The equation $5+3=8$ can be explained in terms of the union of disjoint sets of cardinality 5 and 3. The equation $5\times 3=15$ can be arrived at in terms of repeated addition. Our first encounter with negative numbers tends to be associated with "take away" and equations such as $5-3=2$.

Given such equations, the need for negative numbers arises in seeking a solution to problems such as $3-5=?$. Even here, the solution $3-5=-2$ uses the symbol "–" in two different ways. While there have been efforts to address this kind of ambiguity by writing $3 - 5 = {}^{-}2$, such refinements tend not to be strictly imposed.

One way of bringing order to this situation is to associate the integers

$$\mathrm{I}=\{\ldots,-3,-2,-1,\ 0,\ 1,\ 2,\ 3,\ 4,\ldots\}$$

with the subtraction problems that give rise to them as solutions. But in light of the corruption that the "minus sign" has undergone, let us agree to use a vertical bar "|" in place of the short horizontal bar "–" to which we have become accustomed. In such a system, the integer "negative two" can be written as $3 \mid 5$ or $4 \mid 6$ or $17 \mid 19$ or any ordered pair of whole numbers in which the second is two greater than the first.

By way of making this notation intuitive, we can think of an integer $N = A \mid B$ as the net worth of a portfolio with assets A and liabilities B, where A and B are whole numbers. Such an interpretation may also lead us to use an equal sign to express the equivalence of two portfolios. More generally, it may lead us to write $3 \mid 5 = 4 \mid 6 = 17 \mid 19$ on the grounds that these very different looking symbols all represent the same value "negative two."

In the context of such an interpretation, it becomes reasonable to define the addition of integers as corresponding to the combining of portfolios. That is, given integers $M = A \mid B$ and $N = C \mid D$, the sum of M and N would be defined as

$$(1) \qquad M + N = A \mid B + C \mid D = (A + C) \mid (B + D) .$$

In a similar vein, the subtraction of integer N from M would correspond to the holder of portfolio M being relieved of both the assets and the liabilities in portfolio N. Using the symbol "–" to represent subtraction (and only subtraction!) we arrive at

$$(2) \qquad M - N = A \mid B - C \mid D = (A + D) \mid (B + C) .$$

Using this notation and applying the above rule, the equation $3 - 5 = -2$ would correspond to

$$7 \mid 4 - 8 \mid 3 = (7 + 3) \mid (4 + 8) = 10 \mid 12 .$$

At this point a subtle question arises. Having fallen into the habit of writing $9 \mid 6 = 7 \mid 4$, can we actually substitute one for the other? For example, is it also the case that $9 \mid 6 - 8 \mid 3 = 7 \mid 4 - 8 \mid 3$? Applying (2) to the question at hand, we find that

$$9 \mid 6 - 8 \mid 3 = 12 \mid 14 \text{ and } 7 \mid 4 - 8 \mid 3 = 10 \mid 12 .$$

Since we also write $12 \mid 14 = 10 \mid 12$, it appears that we are free to substitute equivalent representations of integers in applying (1) and (2). Of course this assertion needs to be established in general rather than illustrated in specific cases.

Finally, it remains to deal with the concept of opposite. Having reserved the symbol "–" to denote subtraction, let us use $\left(A \mid B\right)^{i}$ to denote the opposite (or additive inverse) of $A \mid B$. Defining

$$(3) \qquad \left(A \mid B\right)^{i} = B \mid A ,$$

we have $A \mid B + \left(A \mid B\right)^{i} = C \mid C$, where $C = A + B$ and $C \mid C$ can be interpreted as a portfolio of zero value.

Armed with this new machinery, the original problem of evaluating $-(-3-5)$ can be written

$$(4 \mid 7 - 8 \mid 3)^{i} = ? .$$

Applying rules (1)–(3), we obtain

$$(4 \mid 7 - 8 \mid 3)^{i} = ((4+3) \mid (7+8))^{i} = (7 \mid 15)^{i} = 15 \mid 7 = 8 \mid 0 .$$

Here there is no ambiguity of sign. The symbol "–" has been used *only* to represent the operation of subtraction, as defined by (2).

Benefits of Structural Integrity

So what is the value of such an exercise? While we may not want to impose this machinery on children, it does provide a basis for reflecting on pedagogical devices that teachers might be encouraged to bring to bear. For example, the Common Core Standards ask that students

> Understand that positive and negative numbers are used together to describe quantities having opposite directions or values (e.g., temperature above/below zero, elevation above/below sea level, credits/debits, positive/negative electric charge); use positive and negative numbers to represent quantities in real-world contexts, explaining the meaning of 0 in each situation. (NGA & CCSSO, 2010, p. 43)

While temperature does provide a context in which negative numbers are commonly used, its properties are remote from the coherent structure described above. For in what sense is −4° C the opposite of 4° C? Is −8° "twice as cold" as −4°? Given the existence of Celsius, Fahrenheit, and Kelvin scales, is temperature really a likely context in which to "explain the meaning of zero"?

By contrast, credits/debits (assets and liabilities) do seem useful in developing a coherent understanding, at least for older children. For younger children, the idea of positive/negative electric charge (Battista, 1983) can be developed in a non-electrical context, one in which −2 is expressed as

$$3 \mid 5 = + + + - - - - - .$$

Here it is plausible to assert that appending or eliminating pairs of the form $+-$ does not change the overall charge, so that $3 \mid 5$ is equivalent to $4 \mid 6$, is equivalent to $10 \mid 12$, etc. Given this convention, it becomes possible to explain the subtraction of an integer N from M as "a take away problem" by choosing representations $M = A \mid B$ and $N = C \mid D$ in which $A > C$ and $B > D$. For example, one can arrive at the solutions of $3 - 5 = ?$ as

$$\begin{aligned} 9 \mid 6 - 7 \mid 2 &= \left[+ + + + + + + + + - - - - - -\right] - \left[+ + + + + + + - -\right] \\ &= \left[+ + - - - -\right] = 2 \mid 4 \end{aligned}$$

and of $2-(-4)$ as

$$8|6-1|5=[++++++++------]-[+-----]$$

$$=[+++++++-]=7|1\,.$$

In this way, a familiarity with the mathematical structure underlying the integers enables us to provide teachers with classroom tools that are both effective and mathematically sound.

From Subtraction to Division

Another benefit of pausing to develop a coherent approach to negative numbers appears in the study of fractions. As was the case with "take away," some whole number division problems do have a whole number as solution—e.g., $12\div 3=4$. However other problems, such as $14\div 3=?$, do not have a single whole number as solution,[1] and it is this situation that leads us to introduce *rational numbers*, aka fractions. The solution of $14\div 3$ is routinely written $\frac{14}{3}$, a symbol that can be verbalized as "fourteen divided by three." In other words, rational numbers *are* denoted by the division problems that give rise to them!

While our development of negative numbers called for highly unusual notation—i.e., replacing the corrupted subtraction sign "–" by a vertical bar "|", in the case of division we routinely ask students to do essentially the same thing. Even though the division sign "÷" has not been corrupted, our notation for fractions calls for replacing it by a horizontal bar "—" called a vinculum. Furthermore, we ask children to accept assertions such as $\frac{14}{3}=\frac{28}{6}$, even though the two expressions are clearly not the same.

Given these new kinds of numbers as solutions to whole number division problems, there arises a need to extend the operations =, –, ×, ÷ in a credible way. Here the teacher faces the challenge of giving sense to the rules

$$\frac{A}{B}\pm\frac{C}{D}=\frac{AD\pm BC}{BD},\qquad \frac{A}{B}\times\frac{C}{D}=\frac{AC}{BD},$$

$$\frac{A}{B}\div\frac{C}{D}=\frac{AD}{BC},\qquad \text{and}\qquad \left(\frac{A}{B}\right)^{-1}=\frac{B}{A}.$$

These are rules for which the Common Core Standards again emphasize a number line interpretation.

Multiplication of Integers

Not addressed so far has been the multiplication of integers and a coherent way of arriving at $M\times N$ when $M=A|B$ and $N=C|D$. At a pre-algebra level it seems natural to begin with multiplication by a whole number $K>0$ and the rule

$$K\times A|B=KA|KB$$

which can be made plausible in terms of the amalgamation of K identical portfolios of the form A|B. An alternative is to defer such matters until algebraic tools can be brought to bear. Then, recalling that $A|B$ and $C|D$ were intended to represent the solutions of $A-B$ and $C-D$, respectively, it becomes natural to use

$$(A-B)\times(C-D)=AC+BD-(BC+AD)$$

to define

$$A|B\times C|D=AC+BD|BC+AD\,. \qquad (4)$$

Given (4), we would arrive at $-3\times 7=-21$ by writing

$$2|5\times 10|3=(20+15)|(6+50)=35|56=0|21\,.$$

The definition (4) also allows us to arrive at properties such as

$$(-1)\times K=-K$$

that link the integers to multiplication in rather sophisticated ways.

Finally, it is interesting to speculate on whether the relative complexity of the multiplication rule for integers

$$A|B\times C|D=AC+BD|BC+AD$$

is related to the relative complexity of the addition rule

$$\frac{A}{B}+\frac{C}{D}=\frac{AD+BC}{BD}$$

for fractions. After all, our integers were created to deal with the inverse of addition while fractions were associated with the inverse of multiplication. In realms other than the one in which they were created, these numbers become like swans out of water, workable but rather awkward.

References

Battista, M. T. (1983). A complete model for operations on integers. *Arithmetic Teacher, 30*(9), 26–31. Retrieved from http://search.proquest.com/docview/63525541?accountid=14505

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics.* Washington, DC: Author.

Kreith, K., & Chakerian, G. D. (1999). *Iterative algebra and dynamic modeling: A curriculum for the third millennium*. New York: Springer.

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[1] Of course 14 ÷ 3 = ? has the whole number solution Q = 4 and R = 2, which is sometimes written 4R2. In this sense, division is more elementary than subtraction. That is, we have no device for solving 3 − 5 = ? in a whole number context.