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Pokémon Battles as a Context for Mathematical Modeling

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ABSTRACT In this article I explore some of the underlying mathematics of Pokémon battles and describe ways that teachers at the secondary level could explore concepts of mathematical game theory in this context. I discuss various ways of representing and analyzing a Pokémon battle using game theory and conclude with an example of applying concepts of expected value and solving systems of linear equations to find a mixed-strategy Nash equilibrium.

KEYWORDS *Game theory, math modeling, secondary math, classroom activity*

Introduction

There is no denying that games can stimulate students' interest and motivation in the mathematics classroom at any grade level (Bragg, 2012; Devlin, 2011). A more pressing concern is considering whether a game allows for an authentic exploration of relevant mathematics or simply provides a context for loosely connected problem solving opportunities. Oldfield (1991) described a mathematical game as an activity that involves a challenge against an opponent, is governed by a set of rules, has a distinct finishing point, and has specific mathematical cognitive objectives. A mathematical game is more likely to be conducive to achieving its intended learning goals if it requires the use of reasoning that is explicitly tied to the desired mathematical objectives of the game. In this paper I use game theory to explore Pokémon battles as mathematical games with a primary objective of helping students develop the practice of mathematical modeling, which is "the process of choosing and using appropriate mathematics and statistics to analyze empirical situations, to understand them better, and to improve decisions" (CCSSM, 2010, p. 72). These games provide a context to initiate classroom discussions and to utilize various representations in order to analyze decision-making mathematically. Students have the opportunity to apply knowledge of linear equations, probability, and expected value.

Pokémon Background

Pokémon originated as a video game, created by Satoshi Tajiri in 1995 and released in the U.S. by Nintendo in 1998, but the franchise has since extended to include trading cards, anime, movies, and more. The continued pervasiveness of the Pokémon franchise over the last two decades indicates its potential to stimulate student interest in the classroom. The recent success of the *Pokémon Go* app as well as *Pokémon Sun and Moon* indicates a resurgence in the popularity of the Pokémon franchise, especially among high school and college students.

In the original video games, the gamer plays the role of a Pokémon trainer, who sets out on an adventure to capture Pokémon ("pocket monsters") and train them to fight other Pokémon in battles. Pokémon battles are often determined by *type matchups.* There are 18 different Pokémon *types,* such as Fire, Water, Grass, Electric, etc. All Pokémon types come with strengths and weaknesses. It is important to understand the strengths and weaknesses of each Pokémon type because, controlling for some of the other variables in a Pokémon battle, the type often determines the outcome of the battle. Using this simplifying assumption, these Pokémon battles can be modeled with mathematical game theory at a level accessible to secondary students.

Game Theory Background

Game theory is an area of mathematics (but often studied by economists) that is mostly concerned with analyzing how people make decisions. A *game* consists of at least two players who, under a given set of rules, have to make decisions between a specified (in our case, finite) number of actions available to them. An *outcome* results from a specific combination of the players' decisions. The outcome usually involves a *payoff* to each player, which is typically represented by a numeric value that is positive if the outcome is desirable, negative if it is undesirable, or zero if it is neutral.

Analyzing a game with game theory involves determining how each player would make decisions based on certain assumptions about how they behave (e.g., to maximize their own payoff). In order to analyze the decision-making process of the players, it is assumed that all players have *perfect information* of the game, which means that every player knows what the other players would do in any contingency. Because of this perfect information, each player can develop a *strategy*, which is a collection of all the actions that one player would take in response to all of the possible actions made by the opposing players. See Taylor et al. (2009) for more on basic game theory concepts, definitions, and additional examples.

Pokémon Battle Games

At the beginning of the original Pokémon video games, the game's protagonist, Ash, meets with a local Pokémon researcher, Professor Oak, who gives Ash the option to select his first Pokémon from among three options: Charmander (Fire-type), Squirtle (Water-type), and Bulbasaur (Grass-type). The conundrum is clear; Fire-types are strong against Grass-types, which are strong against Water-types, which are strong against Fire-types (much like Rock, Paper, Scissors). Upon selecting his first Pokémon, Ash's rival, Gary, shows up and selects whichever Pokémon is strong against the Pokémon Ash selects (what a cheater!), and then he challenges Ash to a battle. This scenario provides a context for a game. In order to avoid confusion between the video game, the classroom game, and the mathematical game, I will try to be explicit about which game I am referring to. The primary difference between the mathematical game and the classroom game is that the classroom game will involve multiple iterations (e.g., best of seven) of the mathematical game, whereas the mathematical game is technically a static game that is played only once.

Rules of the Pokémon Battle Classroom Game

The Pokémon Battle game can be played between two Pokémon trainers, Ash and Gary, each of whom has a specified number of Pokémon. In early examples, the instructor can provide the Pokémon to be used in the game to ensure that certain concepts can be addressed, but the students can also be allowed to pick their own Pokémon as an extension activity. The game consists of an odd number of rounds, and the objective is to win more rounds than your opponent (e.g., best of seven).

During each round, the battling trainers each select a Pokémon by laying down a card. The teacher can print out "cards" that have a picture of the Pokémon, its name, and its type (or you can use the original trading cards if you have them). Each trainer knows which Pokémon the opposing trainer has, but does not know which one will be selected in a given round. The winner of a round is determined by which of the two selected Pokémon would win in a one-on-one battle—this may be decided in a number of ways. The simplest approach is to use the type matchup; e.g., a Water-type would defeat a Fire-type because its attack does twice the damage. One drawback is that some types are equally matched against others—all of these battles would end in a draw. With this approach, the game is essentially no different than Rock-Paper-Scissors (and therefore possibly not worth all this extra effort). An alternative approach is to have students use probability to determine the winner—this approach will be discussed after the simpler example is used to introduce some notation, terminology, and representation.

Matrix Representation of a Pokémon Battle Game

The Pokémon Battle game can be analyzed as a mathematical game via a matrix representation (see Table 1), which is used in playing the game as well as analyzing it. Each row of the matrix corresponds to one of Ash's possible actions, and each column corresponds to one of Gary's actions. The entry in each cell describes the *outcome* of the battle between the two Pokémon in the corresponding row and column. When students play the game in class, the payoffs can be represented by W for a win, L for a loss, and D for a draw. Later, when the game is analyzed mathematically, payoffs will have numerical

Table 1

Payoff matrix for Game 1

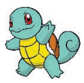
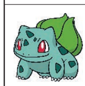
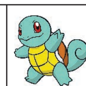
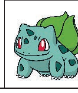
GAME 1		Gary		
		 Charmander (Fire)	 Squirtle (Water)	 Bulbasaur (Grass)
Ash	 Charmander (Fire)	(D, D)	(L, W)	(W, L)
	 Squirtle (Water)	(W, L)	(D, D)	(L, W)
	 Bulbasaur (Grass)	(L, W)	(W, L)	(D, D)

Table 2

Payoff matrix for Game 2 (unfair game)

GAME 2		Gary	
		 Charmander (Fire)	 Growlithe (Fire)
Ash	 Squirtle (Water)	(W, L)	(W, L)
	 Bulbasaur (Grass)	(L, W)	(L, W)

values, but these numeric payoffs may seem arbitrary to students at first. Give students a blank table, and let them fill in the Pokémon names and each of the payoffs for the various outcomes. Each outcome is represented by an ordered pair, where the first entry is the payoff for Ash and the second is the payoff for Gary.

As students play the game, they might start to notice the tendencies of their opponents, which could lead to interesting and insightful discussions. Maybe Gary tends to pick Charmander more often than the other Pokémon. How should Ash respond? Clearly, Ash could choose Squirtle more frequently to capitalize on Gary's tendency to choose Charmander. But, if Ash chooses Squirtle too often, Gary will notice and change his strategy accordingly. Using game theory, we can examine this problem and try to avoid the "But, he doesn't know that I know that he knows…" type of circular reasoning.

Foundations for Analyzing a Game

To simplify the mathematics that will be developed in the following examples, I will focus on two-on-two Pokémon Battle games. One property of the previous example (Table 1) was that there did not seem to be a clear strategy that would allow one of the players to win under all circumstances. Students might appreciate this observation more once they encounter a game in which one of the players has a strategy that allows them to win every round. The example in Table 2 shows a player with two Fire-type Pokémon battling against a player with a Water-type; the player with a Water-type Pokémon has the clear advantage.

Analyzing Game 2 from Ash's perspective, we can think about how he should respond to Gary's potential strategies. If Gary decides that he will play Charmander, he is employing what is called a *pure strategy* (as opposed to a *mixed strategy,* which will be discussed in the next example). In this case, Ash sees that his *best response* to Gary's move is to pick Squirtle. Similarly, if Gary decides to choose Growlithe, then Ash will observe that his best response is again Squirtle. Because Ash's best response to Gary's move in either case is Squirtle, Ash's pure strategy of choosing Squirtle is called a *dominant strategy.* This means that Ash can guarantee himself a better payoff than his opponent, regardless of what his opponent does.

After a few rounds of this game in the classroom, students may start to discuss how this game is unfair. In fact, a good way to introduce this problem is to ask students to decide which trainer they would want to play as. Once students see an example of an "unfair" game, they can begin to discuss what aspects of the game would make it either fair or unfair. Students may begin to see that, in a "fair" game, both players should have some chance of winning, or at the very least, of forcing a draw. Students might consider Game 3 to be fair (see table 3). What characteristics of this game would they think make it fair?

In order to analyze these games prior to introducing numerical values as payoffs, the instructor can introduce students to the idea of representing decision-making with a directed graph (also called a digraph or arrow di-

Table 3

Payoff matrix for Game 3 (fair game)

GAME 3		Gary	
		Charmander (Fire)	Bulbasaur (Grass)
Ash	Squirtle (Water)	(W, L)	(L, W)
	Paras (Bug)	(L, W)	(W, L)

agram) representation. Figure 1 shows a digraph for Games 2 and 3. The nodes in the digraph are the possible outcomes, and the arrows represent instances in which one of the players would be better off changing their action. For example, the digraph for Game 3 tells us that if Ash chooses Squirtle and Gary chooses Charmander, then Gary would be better off if he switched to Bulbasaur. If Gary switches to Bulbasaur, however, Ash would be better off switching to Paras, and the cycle continues. Can students come up with examples of games with digraphs that are different from these?

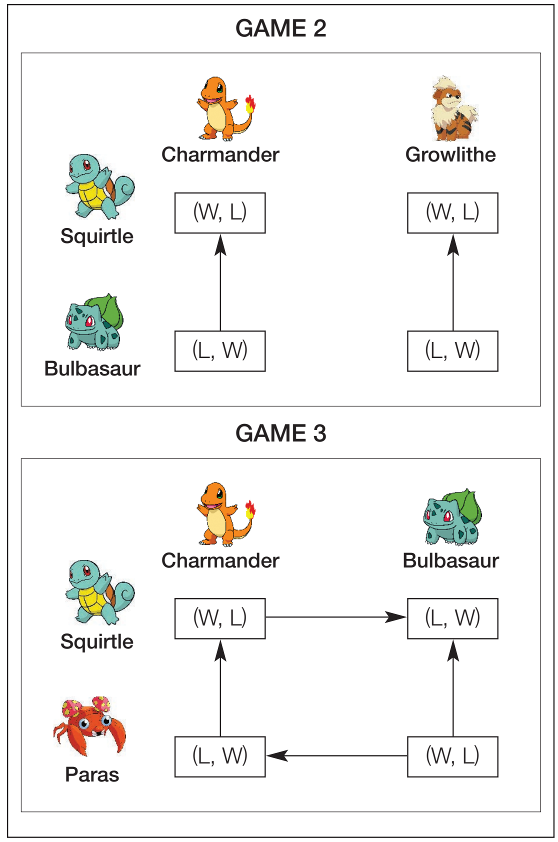


Figure 1. Directed graph (digraph) representations of Game 2 and Game 3.

Numerical Payoffs and Expected Value

One of the underlying assumptions made so far is that the outcome of a Pokémon battle is determined by the type matchup only. As students may point out, many factors can influence the outcome of a Pokémon battle in the video games. A simple way to assign numerical values to payoffs is to let students vote on which Pokémon would win in each matchup. Using the percentages of votes as the payoffs allows students to interpret each payoff as a probability that the Pokémon will win. The payoff matrix in Table 4 shows how a group of students might vote for which Pokémon would win their respective battles. These numerical representations of payoffs set the stage for exploring the concept of expected value.

After the students play the game a few times, the instructor can pose the following situation to analyze the game further. Suppose that the game is played repeatedly, and Ash notices that Gary plays Pikachu more frequently than Charmander, say 60% of the time. Gary is playing what is called a *mixed strategy,* which means that he will play one of his two options according to a pre-assigned probability. Students can discuss how Ash

Table 4

Payoff matrix for Game 4 (numerical payoffs)

GAME 4		Gary	
		Charmander	Pikachu
Ash	Squirtle	(0.8, 0.2)	(0.2, 0.8)
	Bulbasaur	(0.2, 0.8)	(0.5, 0.5)

should adjust his strategy before analyzing the situation mathematically. What are the chances that Ash would win if he plays Squirtle? Since there is a 60% chance that Gary will use Pikachu and a 20% chance that Squirtle would win, the probability that Ash would win is 20%

of 60%, or 12%. On the other hand, there is a 40% chance that Gary will play Charmander and an 80% that Ash would win that matchup; as a result, the probability that Ash would win is 80% of 40%, or 32%. Altogether, Ash's probability of winning is 44% if he chooses Squirtle. This is the *expected value* (an average, weighted by the probability that each event occurs) of Ash's payoff when playing Squirtle. A similar calculation shows that, by playing Bulbasaur, Ash's probability of winning is 38%. So, it seems that Ash is better off choosing Squirtle.

However, in the context of game theory, the assumption of perfect information means that Gary knows that Ash would make these calculations and come to this conclusion. Ash might be tempted to think that his best option is to play Squirtle every time since that strategy gives him the best chance of winning, but Gary would know Ash's strategy and adjust his own in order to improve his own payoff at Ash's expense. Is it possible for Ash to find a strategy that allows him to maximize his payoff without allowing Gary to learn his strategy and take advantage?

Finding a Mixed-Strategy Nash Equilibrium

Ash can potentially improve his *expected payoff* if he responded to Gary's mixed strategy with a mixed strategy of his own. Let p be the probability that Ash assigns to playing Squirtle. Consequently, the probability that Ash assigns to Bulbasaur must be $1 - p$. Then, the following two equations for $E_A(C)$ and $E_A(P)$ give Ash's expected payoff when Gary plays the pure strategies of Charmander or Pikachu, respectively:

$$E_A(C) = 0.8p + 0.2(1 - p) = 0.6p + 0.2$$

$$E_A(P) = 0.2p + 0.5(1 - p) = -0.3p + 0.5$$

Students can analyze these lines either graphically (see Figure 2) or algebraically, but what does the intersection represent? A point on both lines represents a mixed strategy (i.e., probability assignment) that gives Ash the same expected value (i.e., chance of winning), regardless of his opponent's decision. What is special about this expected value being independent of the opponent's decision is that, in a sense, Ash can guarantee himself a certain expected payoff without having to worry about whether Gary will take advantage of it. In this case, the mixed strategy for Ash is to play Squirtle with probability $p = ⅓$, which gives Ash an expected value of 0.4. In the last example, Ash had a 44% chance of winning (expected value of 0.44) by playing the pure strategy of Squirtle. So, has he really improved his situation by using this mixed strategy? To address this question we should consider what would happen if Ash diverged from using the mixed strategy with $p = ⅓$.

If Ash decided to play a different mixed strategy, say with $p > ⅓$, then his expected payoff could either get better or worse, depending on what Gary does (Figure 3). Hypothetically, if Gary observes that Ash is now playing Squirtle more often (increase in p), then he will respond by playing Pikachu more often. When he does this, Ash will notice that he is starting to lose more often and will consequently reduce the amount that he uses Squirtle (decrease in p). In this sense, the mixed strategy represented by the intersection of these two lines is a natural place for the two players to settle.

The point of intersection that we found in this example is called a *mixed-strategy Nash equilibrium* of the game, named for the mathematician, John Nash, who proved that such equilibria exist under certain conditions. Informally, a Nash equilibrium occurs when none of the players has an incentive to unilaterally alter their strategy. If Ash changed his strategy from this one, he

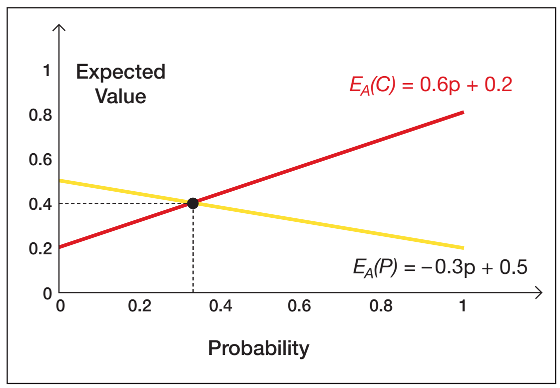


Figure 2. Graphical solution for the system of linear equations.

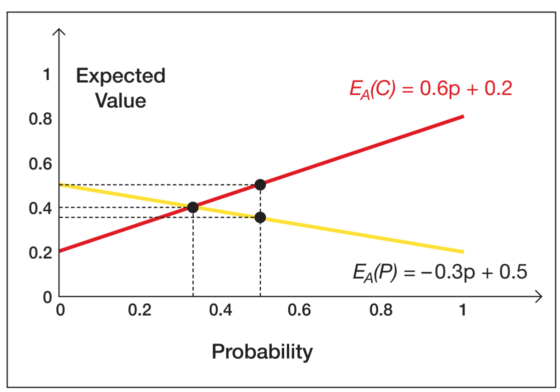


Figure 3. The effect of Ash using a different mixed strategy.

would potentially be worse off than before. Once students find the mixed strategy equilibrium, they can try playing the game with those strategies to check to see if they do better or worse than they did before.

Discussion

Besides demonstrating that a familiar game can be described using relatively accessible mathematics, discussing Pokémon battles in a game theory context has the potential to let students develop their mathematical modeling skills (CCSSI, 2010). The model that we used to represent Pokémon battles certainly has limitations, but these limitations only open the door for students to find ways to improve the model. For example, we determined the payoffs primarily by type matchups and by allowing students to vote on the winner, but the model can be improved by taking other factors into consideration, such as the Pokémon's stats (Attack, Defense, health, etc.). Students might improve the model by finding more meaningful ways to determine the probability of a Pokémon winning a given matchup. On the other hand, a Pokémon battle in the video game is not decided between one matchup but rather lasts until one of the trainers runs out of Pokémon. Students can be given the chance to explore this variation in the game, possibly developing new representations to depict possible outcomes (e.g., tree diagrams) and making additional assumptions (e.g., each Pokémon can only be used in one matchup during a game) to simplify the model.

Pokémon battles can provide a fun context in which students can discuss mathematics, explore new concepts, or refine their skills with familiar concepts. Using game theory in a secondary classroom can also help students develop desirable mathematical practices, such as those outlined in the NCTM's (2000) Process Standards of Connections and Representations. Pokémon Battle games can be used as a starting point for students to examine decision-making in general, leading to the potential to explore connections between mathematics and economics (and other social sciences), biology, engineering, political science, and other fields in which game theory is studied. Moreover, students can be exposed to a wide variety of representations—from payoff matrices and directed graphs to the graphical representation of the relationship between a mixed strategy and the expected value. Allowing students to use their knowledge of a familiar context to develop and improve a mathematical model has the potential to help them understand that they are capable of evaluating a mathematical model, analyzing whether it makes sense according to their knowledge of the situation being modeled, and taking a more active role in their learning of mathematics by posing suggestions for improvement.

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