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Exploring Students' Geometrical Thinking Through Dynamic Transformations Using 3D Computer-Based Representations

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ABSTRACT This paper examines Grade 6 students' thinking about geometrical solids and their properties in the context of a computer-based task. The main focus is to explore how students interact with a number of tasks based on the dynamic transformation of computer-based representations of 3D solids in a geometry classroom with the aid of a Dynamic Geometric Environment. The analysis of two teaching experiments suggests that the dynamic transformations of geometrical solids encourage the students to investigate relationships between the solids and their properties and their 2D representations. Through the tasks, students realize the existence of non-trivial conventions in 2D representations and become aware that angles and edge dimensions may be distorted. The study provides insight into student learning around geometric solids and their properties, and consequently an opportunity to enhance the teaching of 3D geometry.

KEYWORDS *3D geometry, computer-based representations, geometric solids*

Introduction

Students' difficulties in conceptualizing three-dimensional (3D) solids (Ma et al., 2009; Pittalis & Christou, 2010, Sack, 2013) are often attributed to ineffective geometry teaching (Battista, 2007; Ho & Lowrie, 2014; Accascina & Rogora, 2006; Markopoulos et al., 2015a). Ineffective teaching of geometry can sometimes be characterized by procedural pedagogical approaches that merely focus on identifying the properties and definitions of 3D solids without relating the solids' properties to the formation of the solids themselves. Indeed, many students appear not to have the required spatial reasoning skills needed to interpret and understand images of 3D transformations (Suselo, Wünsche & Luxton-Reilly, 2022). In cases where the students experiment with three-dimensional models, learning activities are often limited to the recognition of the solids and to the identification of the critical properties that define the solid (Besson et al., 2010; Markopoulos et al., 2015b; Lowrie et al., 2019). In their study evaluating the types of reasoning that students develop in a dynamic geometry environment, Marrades & Gutierrez (2001) also highlighted the contribution of dynamic computer environments in the development of empirical justifications and their transformation into more analytic and abstract ways of thinking.

Geometrical concepts can be taught to students explicitly in the context of developing and modeling specific tasks in, for example, the field of technology education. However, the use of spatial geometry in programming is often neglected and the teaching of 3D geometry in high school geometry is often absent. (Mammana, Micale & Pennisi, 2012). Indeed, a common pedagogical approach is teaching through rote memorization, often limited to the recognition and naming of geometrical solids and their critical properties, while common teaching and learning activities include "static" exploration of 2D plane representations of solids presented in textbooks (Glasnovic, 2018; Gutierrez, 1996).

The Focus

In this paper, we examine a teaching experiment in which Grade 6 primary school students engage in geometrical thinking about solids and their properties. The study provides insight into student learning around geometric solids and their properties, and consequently, reveals an opportunity to enhance teacher understanding for the teaching of 3D geometry. We explore the role that dynamic transformations can play in creating a learning environment that encourages the development of students' ability to focus on the properties of geometrical solids and, importantly, build relationships between these properties and various solids. Specifically, the research question for the study is: what are some of the mathematical ideas about cubes and rectangular prisms elicited by students working with Dynamic Geometric Environments?

In defining the dynamic transformation of a geometric solid, we consider a process where the solid changes its form through the variation of some of its characteristics and the conservation of others. For example, when transforming a cube to a rectangular prism by increasing the length of one dimension, four faces of the initial cube change from a square to a rectangle, whilst two faces are conserved as squares. It is important to give students opportunities to explore the relationships between the properties of the geometrical solids, so that they are able to properly conceptualize the solids. We explore the implementation of tasks based on the dynamic transformation of computer-based representations of geometrical solids in a mathematics classroom and reflect on how students learn 3D geometry.

Methodology

The research methodology was a classroom teaching experiment along the same lines as described by Cobb et al. (2001). The plan called for students to work in groups on a number of dynamic transformation tasks relating to geometrical solids and their properties, using a central digital projector as a visual reference for the class. In Grade 6, students had already been briefly introduced to the recognition of the properties of a cube and a rectangular prism, along with their construction by using the 2D nets of the solids. The data analyzed in this paper were collected from two Grade 6 classes with the students aged 10-11 years old. The teaching experiment involved the teacher acting as a researcher. The teacher-researcher implemented the tasks, observed students throughout the tasks, and intervened intermittently to explore and clarify students' understanding. Two teaching sessions were analyzed, which focused on the learning environment that was created and the possible cognitive changes concerning students' conceptions of a geometric solid.

3D Representational Models

The materials used in the research study were physical models, computer representations and written imaginary tasks. In this paper, we focus on the computer representations and the way they were used by the students during the teaching session. The computer representations were created using Cabri Geometre II software. A number of research studies have investigated the usefulness of this particular software program in promoting children's geometrical thinking (Laborde, 1993; Leung & Lopez-Real, 2002; Mariotti, 2000). It is acknowledged that computer-based materials allow students to move from physical to cognitive actions, and vice-versa. Research shows that 3D dynamic computer representations provide a learning environment with which students can interact and explore a variety of 3D geometrical objects quickly and easily (Leung, 2011; Højsted, 2019).

The computer-based representations of solids in this study consisted of the isometric and oblique projections of a cube and a rectangular prism. By convention, these types of projections are the most commonly used techniques of pictorial projections in textbooks. This is an important consideration in the teaching and learning of geometry because the use of 2D representations of geometrical solids often follow certain norms.

Figure 1 shows both an oblique projection and an isometric projection of the same cube. In both projections, certain properties of the original 3D cube are maintained while other properties are not. In the oblique projection on the left, the congruence of the edges and angles is only maintained for the front and back faces. On the other hand, in the isometric projection on the right, the equality of the edges is maintained but not the equality of the angles.

In the isometric projection in Figure 1, none of the angles of the cube are 90 degrees. The angles around one corner are 120 degrees. In the oblique projection, the angle between the front face and the base of the cube is 45 degrees. The width of the cube can be drawn in proportion 1:2 to the original, as a cabinet projection. This halving of the depth is an effort to maintain the aesthetic proportions of the cube, as any 1:1 maintenance of the dimensional sizes would result in a cavalier projection (Boundy, 1988).

The two representations of the cube (Figure 1) can be transformed into rectangular prisms by varying the dimensions of the solids, as shown in Figure 2. The changes in dimensions can be achieved by the use of the mouse by "dragging and dropping" the corners. Through these changes, a number of the properties of the solids change. The transformation of the solids in the computer environment does not have the same constraints as transformation of physical models, allowing for greater flexibility with the model. In particular, the length of each of the dimensions could vary from zero, when the solid becomes a plane figure, to any desirable length, though the dimensions of the computer screen constrain the variation so that it remains visible. This characteristic provides students with the opportunity to experiment with different forms of geometrical solids of any length.

Figure 1

An Oblique (left) and an Isometric (right) Representation of a Cube

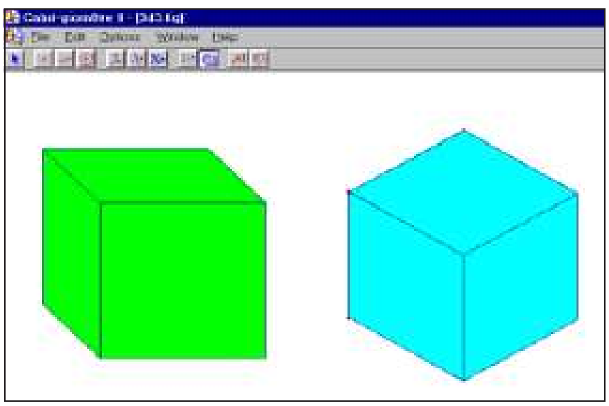
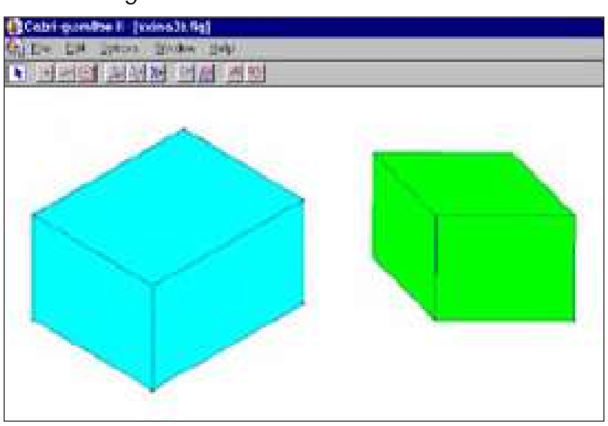


Figure 2

An Isometric (left) and an Oblique (right) Representation of a Rectangular Prism



In the design context, this ability to model and modify virtual shapes is what makes 3D computer representation versatile, as changes can easily be made to the object compared to the production of 3D representations with, for example, pen and paper. The software environment allows students to measure the length of the dimensions and the edges, as well as the size of the faces of the solids, and helps them to interact with the properties of the solids through dynamic transformation processes.

Analysis of the Grade 6 Teaching Experiment

We analyzed two teaching episodes, each from a different Grade 6 geometry classroom. These two teaching episodes were the last ones in the classroom teaching experiment and followed on from three teaching episodes in each of the classrooms where traditional geometric approaches with physical materials were used. The data consisted of field notes and transcribed video recordings. The analysis was a bottom-up process, where issues emerged and were tested during the teaching episodes. From the analysis, a number of issues emerged: the development of the tasks based on the computer representations, the type of interaction between the students and the teacher, and the development of students' thinking about the geometrical solid.

Episode 1

At the beginning of this episode, the teacher-researcher showed an oblique representation of a rectangular prism on the screen and asked students to describe it. The students measured the three dimensions of the solid by using the software and found that the length was 3.4cm, the height was 2.9cm and the width was 2.5cm. Therefore, it was clear that they recognized the solid as a rectangular prism in line with their past teaching experiences. Through follow-up class discussion, the teacher-researcher asked the students to predict which properties would need to be changed in order to transform the solid into a cube. The teacher-researcher used language such as "If you want to transform this solid to a cube what properties of the rectangular prism would you need to change?"

Initially, the students approached the task of transforming the rectangular prism into a cube rather intuitively and transformed the solid by simply decreasing the length of one of its dimensions. Some of the students recognized the new solid as a 'cube', while others thought there was a need to check if the new solid's dimensions were equal.

The teacher-researcher changed the context and asked the students to draw a cube at the board and identify whether the faces were congruent. Students drew an oblique representation of the cube, and tried to justify their opinions:

Teacher-Researcher: *Very well, why is this a cube?*

Student 1: *Isn't this a cube?*

Student 2: *What is it then*?

Student 1: *But we are trying to measure it…*

Student 3: *Despite the fact that it has all the sides equal, in order to be certain, we have to measure them.*

Teacher-Researcher: *Ok, why do you call this a cube? What did you do to get a cube? What did you do first?*

Student 2: *I made two equal faces.*

The teacher-researcher extended the activity aiming to encourage the students to focus their attention on the properties of the cube and to identify the implied drawing conventions:

Teacher-Researcher: *And what are these faces?*

Student 3: *Squares.*

Student 1: *Squares.*

Teacher-Researcher: *Thus, first I make the two faces which are square and then these lines to link the two faces. …*

Student 2: *These (lines) are the edges…*

Teacher-Researcher: *These are the edges. Now the faces now on the side … let's imagine them, as you see them, are they squares?*

Students: *No.*

Student 2: *No, it is a rectangle, no…*

Teacher-Researcher: *Why, why does this happen?*

Student 2: *Because they are oblique …*

Here, the students came to a conflict concerning the concept of the solid. In their previous experiences with the static and dynamic physical models, they had not considered the drawing conventions of the solids. The teacher-researcher introduced the isometric representations as a way to resolve the conflict. He asked the students to recognize an isometric representation of a rectangular prism and then asked them to predict what changes would be required to transform the rectangular prism into a cube. The students correctly identified that a cube must have equal dimensions and began assessing whether the rectangular prism's current edges were all equal. The transformation took place, and a cube with 3.8cm edges was created. The teacher-researcher intervened in the activity and asked the students to justify why the resulting solid was a cube.

The students based their justification on whether the edges were equal, but they did not pay attention to the fact that the angles were not equal. A progression of the students' reasoning was observed. Initially, the students considered the equality of the edges as the only crucial property for a solid to be considered a cube. Next, the students related the cube with the equality of its faces, and lastly, they intuitively attempted to relate the solid with the equality of its angles.

Initially, the students supported the view that the angles were equal. Then they looked for a way to justify their stance. Student 1, for example, proposed to consider the angles of a piece of A4 paper as a model for comparing the angles in the 3D representation of the solid. Student 1 held the corner of the paper up to the screen as a means of comparison. This comparison led the students to conclude that the angles of the solid on the computer were not equal. However, the students still understood that the solid was a cube, although they had discovered that the angles were not equal. Student 1's approach to the teacher-researcher's challenge to justify their opinion was rather intuitive:

Student 1: *We turned around the cube, something like this.* (Showing with her hands an imaginative movement.)

Teacher-Researcher: *Could you explain this to us?*

Student 1: *We turned the cube around to the front and the angles did not appear to be equal.* (She seemed to start to realize conventions.)

Teacher-Researcher: *How?* (He turns a physical model of a cube around in his hands.)

Student 1: *No, we have to turn it …. It is standing on an edge.*

Teacher-Researcher: *This way?* (He turns the physical model around to have an edge in a vertical position.)

Teacher-Researcher: *How does it appear? How do we see it? Like this?* (He kept the cube as an oblique projection where the two faces appear as squares.)

Student 1: *No, turn it, like this ….* (She turned the cube to appear as in the isometric projection.)

Teacher-Researcher: *How? Like this?*

Student 1: *Yes, something like that.*

Students: *Yes.*

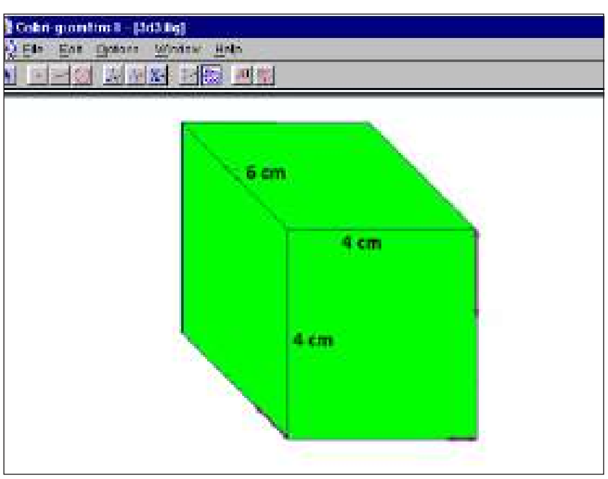
In this exchange, Student 1 was starting to understand the drawing conventions. Initially, she used gestures to assist in explaining her justifications. Later, the teacher-researcher gave her the opportunity to clarify her intuition through the instantaneous use of the computer representation and a physical model. Through the analysis of the previous interaction between the student and the teacher-researcher, it became apparent that the concept of representing 3D geometrical solids and the conventions used to design their plane representations require advanced intellectual processes from students.

Episode 2

In a different classroom, the students were presented with a 3D computer representation of an isometric projection of a rectangular prism. Students were asked to predict the changes required to transform this isometric projection of the rectangular prism into an isometric representation of a cube. The students proposed changing the lengths of the edges, by altering the height and the width of the solid, so as to make it "look like a cube." Apart from varying the dimensions and the edges that corresponded to each dimension in the solid, students made connections between the solid and its angles. They did that by referring to the definition of a cube: namely, that a cube has all lengths equal and all face angles equal to 90 degrees. In their effort to confirm that the solid was indeed a cube, they related the length of its dimensions to the length of its edges and the shape and size of its faces. The dynamic representation on the computer provided the opportunity to measure the length of the edges, the dimensions and area of the faces, and even the volume of the solid (Figure 3). Thus, using this particular strategy, they accurately transformed the solid to a cube.

Figure 3

An Oblique Representation of a Rectangular Prism with measurement functions enabled



Teacher-Researcher: *Are you sure that the (resultant) solid is a cube?*

Student 1: *If you can make it a bit shorter* (pointing in one dimension) *it would look like a cube.*

Teacher-Researcher: *How much shorter?*

Student 1: *A little bit.*

Teacher-Researcher: *Is this enough?*

Student 1: *Yes.*

Teacher-Researcher: *Are you 100% sure.?*

Student 1: *Yep.*

Students: (looking sceptical) *it may be….*

Student 2: *We need to make sure that all the edges have the same length and all the faces are congruent.*

Student 3: *Yes, we need to make sure that all edges have the same length.*

Teacher-Researcher: *How many are the edges?*

(Students are counting the edges).

Student 2: *There are 12 edges. On the screen we can see 9 edges and 3 edges are hidden on the back corner.* [Approach 1]

Teacher-Researcher: *OK. Do I need to check every single one of them?*

Student 2: *Yes, all edges should have the same length in order to be a cube.*

Student 3: *No, we don't need to measure every single one. Only three are enough. One for the length, one for the width and one for the height.* [Approach 2]

At this point, the students developed two different approaches for justifying their assumption that the solid they produced through the transformation was indeed a cube (as noted above). Both approaches required the measurement of certain properties. The first approach was based on relating the cube to the length of its edges, and so they proposed the measurement of this property. The second approach considered the length of the three dimensions of a solid as a key property to justify that the resultant solid was indeed a cube. The main difference between the two approaches was that, in the first case, the cube as the end product of the task was seen as independent of the original solid, while in the second approach, the cube was considered as being derived from the rectangular prism. In particular, the first approach was successful only in the case where the students correctly interpreted the definition of a cube, as a three-dimensional figure, which has 6 square faces, 8 vertices and 12 edges. The students focused only on the equality of the 12 edges of the final solid (a cube) not considering the properties of the original solid (a rectangular prism). They simply tried to make all 12 edges equal in length. On the other hand, the second approach was more constructive, because the students made connections between the original solid and the final solid. In the second approach, students investigated the properties of both solids and identified key properties that link the two solids, such as the lengths of the dimensions and the equality of the face angles. They faced a cognitive conflict that was counter-intuitive. Consequently, they tried to apply the first approach, but they realized that it did not work. Then, using the first approach, they transformed the solid into a cube. The appropriateness of the two methods was negotiated in the classroom discussion, and students realized that the measurement of the lengths of the edges of a cube was adequate.

Finally, the cube was transformed into a rectangular prism by increasing the length of one of its dimensions. During the investigation, students made comparative relationships concerning the variation of the form and the size of the faces of the initial solid (cube) and the final solid (rectangular prism). Moreover, the students compared the length of the edges and the dimensions of the two solids, and ultimately correlated the form of the faces with the length of the dimensions of the solid. While the students initially approached the concept of the solid intuitively, without paying attention to the solid's properties, the use of the dynamic representations allowed them to develop their analytic reasoning. Students built relationships between the properties of the cube and between different solids (the cube and the rectangular prism).

Discussion

The analysis of how the dynamic environment developed in mathematics classrooms highlights the constraints that plane representations of solids pose to students and their thinking. In particular, students, through the study of dynamic transformations, may realize some of the drawing conventions that exist in creating plane representations of solids. In addition, the dynamic transformations of geometrical solids on the computer software encouraged the students to dynamically study geometrical solids and to investigate relationships between the solids and their properties. Moreover, the environment helped students to build hierarchical relationships between different solids, the cube and the rectangular prism. By implementing a dynamic geometry, teachers may be able to improve student outcomes in 2D and 3D geometry. This highlights the importance of developing teacher knowledge in the area of dynamic geometric visualization tools.

It should be noted here that 3D computer representation software may not be accessible to all students. However, schools are gradually becoming more technology focused in the mathematics education environment (Kaput, Hegedus & Lesh, 2007). The increasing availability of educational technology allows for more widespread use of such software.

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