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A Century of Leadership in Mathematics and Its Teaching

Impactful Moments in Mathematics Teaching

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AIMS AND SCOPE

The *Journal of Mathematics Education at Teachers College (JMETC)* is a recreation of an earlier publication by the Program in Mathematics Education at Teachers College, Columbia University. As a peer-reviewed, semiannual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Although many of the past issues of *JMETC* focused on a theme, the journal accepts articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed.

JMETC readers are educators from pre-kindergarten through twelfth-grade teachers, principals, superintendents, professors of education, and other leaders in education. Articles appearing in the *JMETC* include research reports, commentaries on practice, historical analyses, and responses to issues and recommendations of professional interest.

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MATHEMATICS EDUCATION

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PREFACE

In an ideal learning environment, students, teachers, and teacher educators provide mutual support to achieve the common goal of student understanding. The Spring 2024 edition of the *Journal of Mathematics Education at Teachers College* presents three research papers that highlight both how teachers can best support their students and each other, through impactful moments in mathematics teaching. The two short reports explore the theme of impactful teaching moments through the lens of discrete mathematics, one using manipulatives in teaching probability and the other through a problem-based curriculum.

Salami investigates the impact of mathematical game play on student achievement, particularly as a way for teachers to bolster achievement for female students. Through the use of statistical analysis, this research examines the effectiveness of mathematical games for secondary school students in Nigeria, and the gender implications of the results.

Gates and Albert also explore impactful teaching moments, defining them as "mathematical magic." Through qualitative research methods, they illuminate not only the importance of teachers supporting students in these mathematical moments, but also the importance of educators supporting other educators in creating such moments through a mentoring model that pairs early career teachers with faculty mathematicians.

Enu and Ngcobo analyze pre-service mathematics teachers' knowledge and understanding of assessment literacy. Using interviews, the researchers highlight the challenges that pre-service mathematics teachers may face in understanding content knowledge and discuss the recommendations for teacher educators to help with these misconceptions.

Jimmy Giff
Emma LaPlace

Guest Editors

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Effect of Mathematical Games on Senior Secondary School Students' Achievement in Mathematics According to Gender

Olajumoke Olayemi Salami
University of Johannesburg

Erica Dorethea Spangenberg
University of Johannesburg

ABSTRACT Playing mathematical games helps many senior secondary school students—especially girls—acquire basic mathematical skills, but it can be difficult. Thus, this study examined how gender-related mathematical gameplay affects secondary school students' performance. The design of the study was quasi-experimental. Purposively, a sample of fifty senior secondary school students from the Federal Capital Territory (FCT) of Nigeria's Abuja Council Area was chosen. The researcher randomly allocated intact classes to the experimental and control groups using a coin toss. The students' achievement level was determined by an algebraic achievement test administered before and after the treatment. T-tests, means, and standard deviation were used to analyze the data. According to the study, female students had a higher mean score than male students when playing mathematics games. The findings imply that mathematical games enhance mathematics teaching and learning and should therefore be used by teachers to introduce concepts in mathematics to students of all levels, irrespective of their gender.

KEYWORDS *Games, Students, Achievement, Mathematics, Gender*

Introduction

Mathematics is an essential component of educational curricula in most countries. It provides the foundation for logical reasoning, problem solving, and technological innovation. Mathematics is important in many fields, including physics, engineering, economics, and computer science, since it enhances cognitive abilities. Also, Nigeria advocates mathematics as a subject of choice in the secondary school system to prepare students for postsecondary education and to create a workforce for job growth and national development.

However, all parties involved in education, especially mathematics teachers, parents, and the government, are currently very concerned about students' generally low academic achievement in mathematics at this level (Hursen & Bas, 2019). In Nigeria, there is a persistent reporting of students' low mathematics achievement on internal and external examinations. For example, according to the 2019 West African Examination Council (WAEC) report, more than 66% of Nigerian students still need to pass the Senior Secondary School Certificate Examination (SSCE) in mathematics. Additionally, the percentage of SSCE passes with credit and above was 39.5%, 37.5%, 34.2%, and 37.6% in the years 2020, 2021, 2022, and 2023, respectively (WAEC Examiners' Reports, 2020-2023).

Although there are many applications of mathematical concepts in different professions and facets of life, students still perceive mathematics as a challenging subject to learn and a difficult course to pass. The fact that mathematics is usually considered difficult is one of the reasons why students are reluctant to study it in depth (Nicol, 2017). Many students find mathematics complex and scary due to its abstract nature. Students, particularly female students, are inclined to choose not to study mathematics as soon as possible. In addition, many students have negative attitudes toward mathematics, which may significantly impact career choices and contributions to wider society (Ayebale et al., 2020). A strategy to change students' beliefs about mathematics could be mathematical games.

Ogunkola and Knight (2019) opined that mathematical games used in the classroom might positively influence students' academic achievement in science and mathematics. Thus, studies on the impact of games on students' achievement in science have drawn the attention of numerous researchers (Ayebale et al., 2020; Moon & Ke, 2020; Rigelman & Lewis, 2023).

In Nigeria, mathematics games enhance learning by creating an engaging environment and promoting active participation and problem-solving skills (Aduwa, 2021). Research shows that game-based learning positively influences mathematical achievement by providing immediate feedback, encouraging perseverance, and fostering a positive attitude toward mathematics (Akanmu & Adeniyi, 2021). Mathematical games are practical activities that bring about fun, excitement, and challenges between two or more contestants and simultaneously enhance mathematics learning (Khateeb, 2019). Games are mathematical if their strategies, rules, and results are well-defined by mathematical parameters (Asare et al., 2019).

These games often feature straightforward match procedures and rules. Mathematical games like Whot and Ludo show a strong interest in recreational mathematics (Moon & Ke, 2020). In most cases, playing or watching a game is more beneficial than learning arithmetic theory when it comes to understanding the fundamental mathematics of games. Mathematical games aid students in grasping fundamental concepts in mathematics to enhance their arithmetic skills in an engaging way (Rigelman & Lewis, 2023).

Gök (2020) stated that mathematics is entirely game, and games are mostly mathematics. Games share many of the same interactions found in the fundamental structure of mathematics, such as ratiocination, inference-making, and creative thought. In this context, it is convenient to incorporate games into teaching mathematics (Gök, 2020). Incorporating games into the mathematics curriculum is one technique to assist students in learning mathematics while still encouraging fun. The exciting world of games has a positive impact on students' attitudes towards mathematics, enthusiasm to learn, and participation in mathematics lessons. This approach makes learning settings more engaging for students (Moon & Ke, 2020).

Mathematical games can help maintain and sustain student interest, thus leading to good academic results (National Mathematical Centre, 2023). Furthermore, Aduwa (2021) claimed that games are primarily meant for fun and can foster competition and excitement. Games encourage winners to hold on to their lead, and losers to work toward overcoming their loss. According to the National Mathematical Centre (2023), one of the functions of mathematical games is to foster a positive attitude toward mathematics. Positive attitudes can develop because of games' informality and excitement. Exciting activities tend to be more popular with students. Numerous international studies have documented the benefits of using mathematical games in teaching and learning the subject. For example, in Washington D.C., Noah (2019) discovered that mathematical games improve students' comprehension and achievement in the subject. Also, in Pakistan, Shah et al. (2023) found that providing students with opportunities to play mathematical games and receive special instruction helped them reach higher achievement levels in the subject.

Also, the variable of gender was considered in this study. Research on gender and mathematics achievement has yielded various outcomes over time (Bajwa & Perry, 2021). However, current research challenges this assumption, emphasizing the role of social and cultural factors in affecting academic achievement. While some early studies revealed fundamental gender disparities, more recent analyses emphasize the influence of stereotypes and societal expectations in shaping students' self-perception and academic achievement (Maadi, 2022). The natural and cultural phenomenon known as "gender" identifies the different personality traits of men and women, and categorizes in all facets of life (Bajwa & Perry, 2021; Maadi, Mahadi & Alajimi, 2022).

Research on how a student's gender affects their academic achievement has not yielded any clear findings (Korkmazet al., 2023). Therefore, mathematics teachers should constantly incorporate encouragement and reinforcement to pique students' interest in mathematics, particularly female students. This study attempted to determine the effects of mathematical games on the achievement of mathematics in senior secondary school students, particularly in relation to gender in Abuja, Nigeria. The researchers considered the positive impact of mathematical games on mathematics achievement and the fact that many female students do not perform well in mathematics. Thus, this study investigated the impact of mathematical games on the mathematical achievement of senior secondary school students based on gender.

The Impact of Mathematical Games on Students' Mathematical Achievement

Deng et al. (2020) carried out a case study in Shanghai, China, focusing on using digital games for instruction in high school students' mathematics classes. The study's findings demonstrated that students' performance and engagement in mathematics were enhanced when they played digital games once a day for six days. Also, Go et al. (2022) examined how well computer games with mathematical themes support college students' basic mathematical skills. Researchers at a state university in the Philippines carried out a case study that examined undergraduate programs in information technology, mathematics education, and industrial engineering. According to the survey, playing digital games has been shown to help students' basic mathematical abilities. In the study of Sam-Kayode and Salman (2022), it was found that mathematical games help students perform better in mathematics classes and help to develop students' skills.

The study by Khateeb (2019) investigated the impact of mobile gaming on mathematical achievement in Zarqa, Jordan. Their sample consisted of sixty-six fourth graders divided into the experimental group (n = 34) and the control group (n = 32). The experimental group used educational mobile games, while the control group received instruction via the conventional method. The study found that using mobile games to provide students with instructional support in mathematics is an effective strategy.

Prior gender-specific research on how mathematical games affect students' achievement in mathematics in senior secondary school (Sam-Kayode & Salman, 2022) produced mixed results. Some suggest that games encourage participation and problem-solving skills, although methodological limitations and cultural differences remain unsolved. A more rigorous study is needed. This study investigates how mathematical games impact senior secondary school students' academic achievement, with a focus on gender inequalities. It seeks to fill a study gap on the efficacy of games in improving mathematics learning results, potentially leading to more inclusive and effective teaching.

Empirical Research on the Influence of Gender on Students' Mathematical Achievement

Prior studies on the impact of mathematical games on students' achievement in mathematics in senior secondary schools, taking gender into account, have revealed a mixed picture. While some studies suggest that mathematical games improve general academic achievement, gender-specific outcomes differ. Several studies have found that interactive and engaging mathematical games benefit both male and female students by encouraging a deeper knowledge of mathematical ideas (Oliweh & Oyem, 2022). However, a subgroup of studies reveals that the degree of impact varies by gender, with some games being more effective for one gender than the other (Buser & Yuan, 2019). Overall, the convergence of previous studies highlights the necessity of taking gender dynamics into account when applying mathematical games in educational contexts, enabling a nuanced and inclusive approach to improving learning results.

Due to the contradictory nature of research findings on gender and mathematics, teachers continue to disagree notably about the impact of gender on students' achievement in this subject (Liang et al., 2020; Liu & Hwang, 2020; Akanmu & Adeniyi, 2021; Alsadoon et al., 2022). Singh et al. (2021) compared the average achievement ratings of male and female students who learned numbers using mobile games. The findings demonstrated that male students outperformed female students when taught with mobile gaming.

Ellison and Swanson (2023) used competition data to investigate the dynamics of the gender gap in high school mathematics achievement. By ninth grade, there is a noticeable gender difference, and it gets bigger with time. The gender gap is getting more comprehensive as a result of gender-related variations in dropout rates, as well as in the mean and variance of year-to-year improvement. According to the study, only some girls achieve significant enough gains to raise their rankings.

Buser and Yuan (2019) used regression discontinuity analysis to study the Dutch Mathematics Olympiad participants on the verge of moving on to the next round. For boys, they find a small and insignificant dropout effect of one percentage point, and for girls, a large and marginally significant dropout effect of eleven percentage points. We can use narrower windows and obtain more accurate estimates due to our significantly larger sample. The study discovered that girls are comparatively likelier to respond by quitting, but we also show that boys are greatly affected. The difference in effect between girls and boys in the competition is less significant than their point estimates for the Netherlands.

Oliweh and Oyem (2022) focused on the gender disparities in students' mathematics achievement in a selected secondary school in Delta State, Nigeria. The method of stratified random sampling was applied.

There were eight hundred secondary school students in the study population. To direct the investigation, a single research question and hypothesis were developed. Data were gathered and examined according to the study's objectives using the Statistical Package for the Social Sciences (SPSS). The hypotheses were tested at the significance level of 0.05. The mean variance of the research hypothesis was examined using the t-test analysis. The model's t-test p-value served as the basis for the decision. Levene's test for equality of variance supports the homogeneity of the variance. The hypothesis's findings demonstrate no difference between male and female students' math achievement.

Anokye-Poku and Ampadu (2020) conducted a study to investigate how girls and boys in Ghanaian junior high schools felt about mathematics and their success in the subject. A sample of 360 students was used in a descriptive survey design. Two tools were used to evaluate the student's performance: test results and a semi-structured questionnaire. The findings showed a statistically significant achievement gap, with male students outperforming female students in mathematics.

Theoretical Framework for the Study

Two theories – cognitive development and behavioral learning – provided the foundation for this study, which employed mathematical games. On the one hand, playing mathematical games while learning mathematics leads to cognitive development. Specifically, the study employed the cognitive development theory to examine how teachers can use mathematical games to assist learners in conceptualizing mathematics (Slavin et al., 2021). According to this developmental theory, student interaction on relevant tasks improves their grasp of essential ideas (Ekmekci & Serrano, 2022). Students gain a deeper understanding of the material they are studying when they interact with other students and have to explain and discuss each other's points of view (Slavin et al., 2021). According to the cognitive development theory, explaining the subject to someone else is one of the best ways to learn it. Playing mathematical games encourages more elaborate thinking and frequent explanations, improving understanding depth, reasoning quality, and long-term retention accuracy (Dimosthenous et al., 2021; Liang et al., 2020; Deng et al., 2020).

On the other hand, according to the theory of behavioral learning, students are more likely to commit to teamwork if teachers give them rewards for their participation (Morgan et al., 2019). Therefore, rewards for individuals and teams should be clear when employing mathematical contests. Thus, from both of these theoretical perspectives, using mathematical games should result in better student learning.

As a result, this study demonstrated the two main theoretical stances on using mathematical games in education: the cognitive development theory, which emphasizes the effects on students, and the behavioral learning theory, which emphasizes the students' incentives to complete academic work, such as reward and goal structures (Slavin et al., 2021). These theories provide teachers with the framework to design learning environments catering to a classroom's diverse learning styles, interests, and abilities. Adopting the theories mentioned earlier suggests that teachers should break away from traditional methods of instruction and instead implement innovative strategies like audio, video, and field trips to give students more ways to absorb knowledge (Chiang et al., 2019). With the help of these techniques, students can succeed and learn at their own pace and style (Esperanza et al., 2023).

Purpose of the Study

The study aimed to examine how mathematical games affected senior secondary school students' gender-related academic performance in mathematics. The precise goals were to:

- Determine the impact of mathematical game use on students' mathematical achievement.
- Ascertain how gender affects students' mathematical achievement when utilizing mathematical games for teaching and learning.

Research Hypotheses

The 0.05 level of significance formulates and tests the following null hypotheses:

- There is no significant difference in the mathematical achievement of students who were taught through mathematical games compared to their counterparts who were not.
- The effect of gender on students' mathematical achievement when they were taught with mathematical games is not statistically significant.

Significance of the Study

The following groups will benefit from the results and recommendations of this study:

- Teachers will be provided with mathematical games to introduce new concepts and to engage students in learning mathematics.

- The government or the curriculum developers will organize regular training workshops and seminars for mathematics teachers so that their knowledge of mathematical game use in the classrooms can improve.
- Scholars contribute vital evidence and guide future educational practices by providing real evidence to existing knowledge. The report also identifies gaps in current knowledge and suggests directions for future research. This study contributes to the consolidation of existing knowledge, identifies areas requiring additional inquiry, and provides a thorough review for educators, researchers, and policymakers.
- It is vital to highlight that the outcomes of this study will also be used as a reference for other researchers who may want to perform similar studies in other parts of Nigeria.

Limitations

The following limitations are pertinent to this study:

- The number of fourth graders randomly chosen from senior secondary school students in the Abuja Council Area of the Federal Capital Territory, Abuja, constitutes the study's sample size.
- The impact of mathematical games on senior secondary school students' academic advancement in mathematics is impacted by contextual limitations in Abuja, Nigeria, particularly in terms of gender. Sociocultural factors, educational infrastructure, and gender norms may all have an impact on the effectiveness of these activities, making it challenging to attain equitable results for male and female students.
- Tool and content validity are limited, due to the reliability of the instruments used in this study.

Methodology

Research Design

This study used a quasi-experimental design. The two groups—one experimental group and one control group—were instructed in using mathematical games. While the control group received instruction using the standard methodology, the experimental group was exposed to treatments utilizing an instructional strategy based on mathematical games.

Pre-testing on the experimental and control groups determined the student's level of achievement, while the post-test after the treatment measured the change in learning outcomes in the groups. In this study, the mathematical game instructional strategy was the independent variable, the student's achievement represented the dependent variable, and gender (male and female) was the moderator variable. Using a pre- and post-test, the researcher was interested in manipulating the independent variable (game) to observe its effect on the dependent variable (achievement in mathematics).

Sample

The sample comprised 50 senior secondary school students in the fourth grade in the Abuja Council Area of Federal Capital Territory, Abuja, selected purposively from a population of 1,086 students. The fourth grade is the foundation class for senior secondary school classes. There were 30 males and 20 females in the sample. Three schools were selected for the investigation using simple random sampling. The schools selected for this study had (1) at least one teacher with a BSc (Ed.) Mathematics qualification, (2) teachers that taught mathematics, and (3) a considerable distance between the other schools. A coin toss was used in each selected school to assign intact classes to experimental and control groups randomly. The experimental group had 16 male and 10 female students, while the control group had 14 male and 10 female students.

Data Collection

Before the experiment, two research assistants received training on conducting the experiments, which took place in one of the schools. The researcher explained the rules of the algebraic substitution game to the two research assistants and provided an outline of the algebraic expressions to be taught to learners. The research assistants taught the students about algebraic expressions over three weeks before the commencement of the experiment.

The study consisted of two stages: The preliminary stage and the implementation stage. During the preliminary stage, the experimental and the control groups were pre-tested using the Algebraic Achievement Test (AAT) before the teaching. The researcher created the 20 items that made up the AAT. Some examples of algebraic expressions are $3x+2y-5$ or $2a^2-4ab+b^2$.

The study explores the impact of mathematical games on senior secondary school students' academic achievement, highlighting the complex interaction with gender dynamics. It suggests that gender-related issues, cultural barriers, and contextual factors, such as educational infrastructure and community attitudes, may influence outcomes. The purpose of the test was to ascertain the students' entry-level behavior.

During the implementation phase, the topic of algebraic expressions, scheduled for two weeks in the mathematics scheme of work and four periods per week on the school timetable, was taught to the experimental and control groups. The researcher provided an example of algebraic expressions, along with guided practice. Variable and verbal expressions were as follows:

(a) Examples of algebraic expressions: (i) x increased by 6 is $x + 6$; (ii) the quotient of 18 and n is 18/n
(b) Examples of verbal expressions: (i) $x/2$ is half of x; (ii) $5n$ is five times a number, etc.

The students were taught algebraic expressions in a 40-minute lesson using eight periods throughout the two weeks. Following every lesson, students in the experimental groups solved an algebraic expression exercise. Students solved questions on algebraic expressions from the New General Mathematics book for senior secondary school (Murray et al., 2008).

Thereafter, the students were screened with the Algebraic Achievement Test (AAT), and those who scored a threshold of 40 or lower were assigned to one of four groups, three experimental and one control. Learners in the three experimental groups played the mathematical game.

The study aimed to examine the impact of a modified mathematics game on students' mathematics achievement in a varied classroom context. A modified version of "Snakes and Ladders" is used to improve mathematical learning by adding mathematical questions to each ladder or snake position on the board.

One group used a collaborative learning model, which promoted teamwork and cooperative problem solving. Students in this group explored the mathematical issues collectively, creating a supportive environment that promoted collective comprehension. Another group engaged in competitive gameplay, with individuals competing to answer challenges and to move up the board. This strategy attempted to create a sense of drive and urgency by imitating real-world problem-solving settings.

A third group utilized self-paced learning techniques, focusing on personalized learning and incorporating interactive technology to enhance understanding of mathematical concepts. The teacher supervised the groups to ensure that they followed the guidelines for solving algebraic expressions as described by Alsadoon et al. (2022), which are as follows:

- Each group of students had to work as a team and reach a decision by consensus.
- Every member of the group impacted the achievement of the others.
- Each assignment received a grade, and each group member received their group grade as their grade.
- Students could seek help from each other towards attaining a common goal.

For the post-test, the items in the pre-test (AAT) were re-arranged. The post-test was administered to all groups at the end of the mathematics game to assess what the students have learned. The tests were marked and graded.

Data Analysis

Descriptive and inferential statistics were employed to analyze the gathered data. The average and standard deviation display the pre-test and post-test results. Secondly, t-test statistics were used at a significance level of 0.05 to test the null hypotheses.

Quality measures

Three experts critically appraised the face and content validity of the instrument (AAT). The outcome of the appraisal of the items based on the experts' judgment gave a 0.85 index of logical validity.

The validated instrument (AAT) was trial tested to determine the reliability of the instruments, using 24 students who were part of the study's target population, but did not form part of the study. Also, the split-half reliability method establishes the instrument's internal consistency with a reliability coefficient of 0.84.

The instrument was considered adequate using psychometric item analysis, with an average of 0.56 as the difficulty index, 0.63 for the discrimination average index, and -0.13 as the distracter average index for the items.

The researchers implemented the following protocols to control for unrelated variables that could introduce bias into the study. The researcher put together a standardized training program for the teachers who worked as research assistants. The same research assistants managed the experimental and control groups. The test instruments were under the researcher's care, and research assistants helped only when asked by the teachers and when needed. The researcher assumed a supervisory role in averting the teachers' departures from the prescribed content of the lesson plans. The researcher obtained the pre-test and post-test that the instructors and students gave.

Ethical considerations

Permission was granted by the ethical committee of the federal university, Oye Ekiti, to conduct this study. The researcher sought permission from the principals of the two selected schools. Consent forms were completed by students in these schools agreeing to participate in the study after discussing the students' consent, objectives, and activities. The researcher ensured that all works cited were referenced and paraphrased. After considering all ethical issues, a plagiarism check was conducted on the study to ensure high originality. The researcher also sought the support of two mathematics teachers in the selected secondary schools who served as the research assistants while administering the treatments to the experimental groups in their respective schools.

Results

Descriptive statistics were first used to show the participants' biographical information, assess the remaining items' internal consistency, and determine the discriminant validity of the AAT with respect to mathematics. Second, the inferential statistics analysis focused on the test statistics of the variables, which included student achievement and gender. The results are shown based on the variations in the cross-variable averages when there were significant differences. It was determined through descriptive statistics that there were more male participants (60.0%; 30 out of 50) than female participants (40.0%; 20 out of 50).

Research Question 1

What influence does teaching through mathematical games have on students' mathematics achievement?

Table 1 presents the experimental and control groups' mean gain scores for mathematical achievement. The pre-test mean scores for the experimental group (M = 41.47, SD = 15.03) and the control group (M = 41.38, SD = 18.13) differ slightly. The post-test mean scores, on the other hand, differed more between these groups (M = 61.12, SD = 15.03 and M = 45.38, SD = 18.13, respectively). The control group's mean gain was 4.00, while the experimental group's was 19.65. This result indicates that the experimental group performed better than the control group in terms of gain score.

Research Question 2

How does teaching through mathematical games influence students' achievement in mathematics pertaining to gender?

Table 2 presents the descriptive statistics on the influence of gender on students' achievement when taught with mathematical games. The female students had higher mean achievement scores (*M* = 45.34, *SD* = 21.72) than the male students' (*M* = 40.02, *SD* = 14.43) for the pre-test before they were taught through mathematical games. However, the mean score between these two groups for the post-test (*M* = 76.35, *SD* =21.72 and *M* = 56.42, *SD* = 14.43, respectively) increased notably after they were taught through mathematical games. Thus, female students outperformed male students.

To establish whether the differences were significant, correlational statistics were employed using a t-test to measure the formulated null hypotheses at $p \leq 0.05$. A t-test is employed to determine if there's a significant difference between the means of two groups, indicating whether observed variations are statistically meaningful or just due to chance.

Table 1

The Experimental and Control Groups' Mean Gain Scores for Mathematical Achievement

Group	N	Pre-test Mean X_1	Post-test Mean X_2	Mean Gain Score X_2-X_1	SD
Experimental	26	41.47	61.12	19.65	15.03
Control	24	41.38	45.38	4.00	18.13

Sources: Authors' computation from SPSS

Table 2

Descriptive Statistics on the Influence of Gender on Students' Achievement when Taught with Mathematical Games

Group	N	Pre-test Mean X_1	Post-test Mean X_2	SD
Male	30	40.02	56.42	14.43
Female	20	45.34	76.35	21.72

Sources: Authors' computation from SPSS

Hypotheses 1

There is no significant difference in the mathematical achievement of students taught using mathematical games compared to their counterparts who were not.

Table 3 presents the independent t-test of the experimental and control groups. The difference between the experimental and control groups on the mathematical achievement of students who were taught through mathematical games. It is shown that the mathematical games of the experimental group had a significant influence on students' mathematics achievement among senior secondary school students [t (48) = 5.28, p=0.0019; p < .05], The results of the independent sample t-test are displayed in Table 3; the t-test value is 5.28, the degree of freedom is 48, and the p-value is 0.0019. The experimental group's mean is statistically higher than the control group's because the p-value is less than the 0.05 significance level. It is therefore asserted that there was a noteworthy distinction in the academic performance of senior school students who were taught mathematics through mathematical games and their counterparts in the control group. This demonstrated that the experimental group's usage of mathematical games significantly influenced their academic performance.

Table 3

The independent t-test of the Experimental and Control Groups

Group	Mean	N	Df	T	P
Experimental	19.65	26	48	5.28	0.0019
Control	4.00	24			
Sources: Authors' computation from SPSS					

Hypotheses 2

The effect of gender on students' mathematical achievement when they were taught with mathematical games is not statistically significant.

Based on Table 4, the result showed that gender had a significant influence on mathematical games among senior secondary school students [t (48) = 3.57 p=0.01; p > 0.05]. Since the p-value is lower than the threshold of 0.05 significant coefficient, female students (mean = 31.01) made progress using mathematical games, compared to male students (mean = 16.40) on mathematics achievement. Therefore, it implies that female students perform better than male students after being exposed to mathematical games.

Table 4

The independent t-test of the Experimental and Control Groups

Group	N	Mean	Df	T	P
Male	30	16.40	48	3.57	0.01
Female	20	31.01			
Note: The correlation, denoted by p, is significant at the 0.05 probability level					

The mean gain scores of the students who participated in the math achievement tests are displayed in Table 4 according to their gender. When teaching mathematics through mathematical games, the mean gain score for female students was (x=31.01), whereas the mean gain score for male students was (x=16.07). Female students' mean gain scores were 14.94 higher when exposed to mathematical games than male students.

At the 0.05 level, both statistical tests assumed a significant correlation: the differences between the male and female variables were statistically significant. Consequently, female students made notable progress when taught mathematics using mathematical games.

Discussion and Conclusion

According to the study, teaching through mathematical games in senior secondary schools significantly increased students' mathematics achievement (*M* = 61.12, *SD* = 15.03 and *M* = 45.38, *SD* = 18.13). The experimental group's mean was (x= 19.65), while the control group's mean was (x= 4.0). The results show that the experimental group of students who played a mathematical game while learning the subject had higher mean scores than the control group of students who did not. Akanmu and Adeniyi (2021) investigated the impact of mathematical games on students' achievement in mathematics and discovered that it helps students perform better in the subject. As a result, compared to students taught without games, students taught with mathematical games produced noticeably better results in mathematics. Students could have achieved better when taught with mathematical games, as mathematical games illustrate theoretical and abstract concepts with concrete examples, which facilitate effective mathematics teaching and learning.

It is also evident from the study that female students outperformed male students, with the female students' mean value (x=31.01) being higher than the male mean value (x=16.40). The games were effective for male students in raising achievement and cultivating a positive attitude toward mathematics, despite their initial lack of enthusiasm. This was confirmed when they were encouraged to play. Mathematical games are valuable learning opportunities for both students and teachers, according to the teachers who oversaw their supervision. This finding corresponds with the findings of Yeh et al. (2019) and Ayebale, Habaasa, and Tweheyo (2020), who affirmed that female students exposed to mathematical games had a mean gain score higher than their male counterparts. Thus, gender significantly influenced students' achievement when mathematics was taught using mathematical games.

Given the identified factors, mathematics teachers should expose both genders to constructivist teaching methods, such as computer game instructional strategies (Noah, 2019), as mathematical games positively influence mathematics achievement. Students should learn by doing, taking an active role in building their understanding. Teachers should utilize mathematical games to help male and female students reach their maximum potential. It is, however, essential to note that female students benefit more from being exposed to mathematical games. Thus, teachers should expose female students to mathematical games that attract their interests.

As lack of interest in mathematics is a notable reason for student failure in the subject, mathematical games for teaching mathematics classes may hold a more significant promise of regaining students' achievement in mathematics, especially for female students.

This study examined how mathematical games affected the mathematical achievement of senior secondary school students in terms of gender in Abuja, Nigeria. According to the results, teaching through mathematical games significantly improved students' achievement in mathematics. Also, female students achieved higher scores than male students when taught through mathematical games.

The findings imply that teaching through mathematical games can enhance mathematics achievement and that mathematical games should, therefore, be used by teachers to introduce mathematics concepts to students at different levels. Mathematical games can create a friendly atmosphere for student-centered teaching and learning to engage students in learning mathematics concepts. By organizing regular training workshops and seminars for mathematics teachers, teachers' knowledge of mathematical games can improve. School administrators should provide teachers with simple local games like Whot, Ludo games, and playing cards to facilitate the teaching and learning of specific topics in mathematics. However, using mathematical games in teaching is inadequate in accounting for continuous achievement disparity between males and females in mathematics across senior secondary schools in Abuja. Other factors such as the students' attitudes, self-efficacy, and ability levels should also be considered.

The study was limited to two senior secondary schools in an urban area in Nigeria. The authors suggest similar research in other contexts. Studies comparing urban and rural senior secondary school students' mathematics achievement when taught through mathematical games are also recommended. The authors further suggest using online mathematical games in longitudinal studies to follow students' progression in learning mathematics.

Parents may use the outcome of this study to provide educational games to their children at home to engage them in mathematics. Such games can provide opportunities for monitoring, advising, and encouraging their children toward positive achievement in mathematics.

Overall, including mathematical games into experimental investigations, together with a variety of gameplay methodologies, proved to be a novel and successful method for improving students' mathematics efficacy and achievement. Curriculum planners may find teaching with mathematical games beneficial when designing programs for mathematics teachers. The government may also appreciate and use the study's findings to improve and implement the general objectives of mathematics education as drawn up by the Federal Republic of Nigeria (FRN).

Teaching through mathematics games may make students more focused and willing to dedicate considerable time to engage in mathematics in the classroom. Teaching mathematics to senior secondary school students through games improves their academic achievement. Hence, teachers need to develop themselves by using mathematical games to enhance students' academic achievement in mathematics.

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Informed Consent

The authors have obtained informed consent from all participants.

Conflict of Interest

The authors declare that there is no conflict of interest.

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Considering Opportunities for Mathematical *Magic:* Design Principles for a Mathematician-Early Career Teacher Mentoring Model

Miriam Gates
Emmanuel College

Lillie Albert
Boston College

ABSTRACT Mathematical *magic*, or experiences of beauty and creativity for K-12 mathematics students, can contribute to deepened content understanding. However, creating opportunities for mathematical *magic* in mathematics classrooms has been a challenge for classroom practitioners. To support development of these experiences in classrooms, we deployed a bi-directional support program between a college-level mathematician and an early-career secondary teacher. Based on data collected from this project, we have identified four principles that will promote mathematical *magic*. We conclude that mathematicians and early career teachers can work together to a) leverage understandings of the nature of mathematics, b) apply meta-mathematical reasoning, c) share knowledge of content and students across contexts, and d) recognize students' mathematical development to design learning experiences for mathematical *magic*.

KEYWORDS: *Mentoring, early-career teachers, mathematics faculty*

[W]hat's really challenging for me is one of the best things that I did for my own education was these number theory classes and these proof writing classes. I feel a lot of the *magic* of it is missing from my classes. You know, the power of looking through data and looking through numbers and coming up with some conjecture, then proving or disproving it. And that intense feeling, although it has been discovered before, in the moment, you feel like you are a genius. Or you are the only one that's ever discovered it. It's almost like a feeling of creation [...] It's how I imagine maybe artists feel sometimes, when they feel like they've made something through their own devices.

– Ms. Jackson, a teacher participant

Experiences of mathematical *magic*, or the creativity and beauty in mathematics, as described by Ms. Jackson, have been associated with positive mathematical identities (Palmer, 2009), increased interest in mathematics (Brinkmann, 2009), and deepened mathematical understanding (Cellucci, 2015). These approaches align with how mathematicians describe their own disciplinary experiences: beautiful, personal processes of doing (Burton, 2007). However, this kind of *magic* is often missing from secondary classrooms, according to teachers (Karp, 2008). At the post-secondary level, some mathematicians describe teaching as procedural communication where students must identify the underlying beauty (Burton, 2007). Therefore, students at the K-16 level do not always experience mathematical *magic*, despite its importance to both teachers and mathematicians. Further, a majority of the research about college-faculty mathematicians in mathematics education has been in their own post-secondary contexts (e.g., Burton, 2001; Nardi, 2008; Sfard, 1998), rather than at the K-12 level, so how they might support beginning teachers is not widely understood. Here, we propose to examine how the dual expertise of a beginning teacher and a professional mathematician in a mentoring relationship might be leveraged to produce mathematical *magic* within secondary schools.

To better understand how such an interaction might look, we completed an exploratory study associated with Exemplary Mathematics Teachers for High Needs Schools (EMTHNS). As part of a larger project, teachers in their first four years of teaching were mentored by both college-faculty mathematicians and experienced secondary mathematics teachers. In this paper, we focus on the teacher-mathematician relationship and propose four principles that leverage the expertise of mathematician and teacher to create a vision for mathematics classroom practice that includes mathematical *magic*. In what follows, we will begin with the relevant literature and describe the complete program in which teachers and mathematicians participated. Based on data analysis, we propose design principles for similar mentoring programs. The concluding section presents implications for the field.

Situating the Challenge of Mathematical *Magic*

This project is situated in the literature about experiences of the aesthetic of disciplinary mathematics and mentoring of early career mathematics teachers. Here we explain how the concept of mathematical *magic* relates to the literature on aesthetics in K-12 classrooms. We will also describe some of what is known about faculty mathematicians in mathematics education research and mentoring early career mathematics teachers.

Considerations of aesthetics in mathematics classrooms

The concept of mathematical *magic*, the appreciation of the aesthetic of mathematics, is important for teachers because it allows them to see themselves in the discipline (Hobbs, 2012). For learners, aesthetic considerations have been shown to support the development of a positive mathematician identity (Palmer, 2009) and to stimulate interest in mathematics (Brinkmann, 2009). Mathematical aesthetic also can provide a route toward a deeper understanding of the discipline (Cellucci, 2015). Additionally, it is essential to mathematical creativity (Brinkmann & Sriraman, 2009) and the judgement of problem posing in mathematics (Crespo & Sinclair, 2008). For professional mathematicians, mathematics is a beautiful, personal process of doing or, in our words, *magic* (Burton, 2007).

At the post-secondary level, for some faculty, mathematics teaching is procedural communication where students leverage procedure to identify underlying beauty (Burton, 2007). Other research centers teacher candidates' development of content and pedagogical knowledge through problem solving and inquiry (Apkarian et al., 2023), developing mathematically-focused art (An et al., 2023), or conjecture-building activities (Meagher et al., 2020) that could lead to mathematics *magic*. Thus, PTs may or may not have been exposed to experiences of mathematical *magic*.

Similarly at the K-12 level, mathematical aesthetics have not always been a central concern (Sinclair & Crespo, 2006). In fact, Karp (2008) argues that teachers suggest beautiful mathematics exist outside of the mathematics curriculum. However, recent studies attempt to better define creativity and determine how to encourage it at the K-12 level (e.g., Assmus & Fritzlar, 2022; Lu & Kaiser, 2021). Furthermore, practices that support *magic* are not sufficient to provide equitable and empowering mathematics experiences for traditionally underserved students because attention to students' cultural backgrounds is critical (Battey, 2013). Therefore, there is space for collaboration to ensure equitable approaches to creativity in the K-12 classroom.

Faculty mathematicians in mathematics education research

Mathematicians target two major ideas in courses for prospective teachers (PTs) to understand: the mathematical content and the meta-mathematical experiences, such as the aesthetics, proof, and uses of mathematical definitions (Leikin et al., 2018). However, a lecture does not necessarily convey these kinds of messages to students, even at the post-secondary level (Lew et al., 2016).

In parallel research, due to their different professional roles, professional mathematicians tend to attend to *what* students learn, while teachers tend to focus on *how* students learn (Wade et al. 2016). Therefore, mathematicians, mathematics educators, and mathematics researchers can use different expertise to support learners (Bass, 1997). Mathematicians involved in mathematics education act at the boundaries to transform classroom practice, in both K-12 and the post-secondary classrooms (Darragh, 2022). This mentoring program seeks to support participating teachers and mathematicians to work synergistically to support mathematical *magic*.

Existing mentoring models

Mentoring and induction programs have supported early-career teachers to stay in the profession, increase scores on classroom practice measures, and improve student achievement (Ingersoll & Strong, 2011). Other benefits for both mentor and mentee are productive relationships with colleagues, reflection on practice or subject matter, professional development, and personal satisfaction (Ehrich et al., 2004). Additionally, field-based mentoring programs can encourage new teachers to change beliefs about groups of students (Garza & Harter, 2016). A constructivist mentoring relationship, rather than transmission-style, supports first-year teachers with their self-efficacy, enthusiasm, job satisfaction, and emotional support (Richter et al., 2013). Reported challenges for mentees and mentors include lack of time, professional or personal mismatch, a non-productive relationship, and additional programmatic commitments (Ehrich et al., 2004).

Research suggests that a content mentor plays a unique role in developing content knowledge and could bolster novice teachers in developing student disciplinary thinking (Achinstein & Davis, 2014; Wang & Paine, 2001). There is limited research on mentoring between professional mathematicians and beginning teachers. In reporting on Math Circles where faculty and teachers worked together, college faculty described extending their pedagogical knowledge and conversations and teachers identified deepened content knowledge as well as expanded repertoires of classroom approaches (Cushman et al., 2023). Further, a mentoring model between PTs and faculty has suggested, among other parameters, that well-established mentoring guidelines and clear communication channels between stakeholders in the project strengthen the program (Hodge et al., 2019). In aligned research on apprenticeship programs that invite science teachers to spend summers assisting scientists an increase in discourse practices, collaboration, changes in classroom practices (Sadler et al., 2010), changes in pedagogical beliefs (Miranda & Damico, 2013), and beliefs about the nature of science (Baker & Keller, 2010) have been identified. In sum, teacher exposure to subject matter experts can positively impact their classroom practice, when appropriately framed. The program described here was designed to benefit to both the teacher and the mathematician as described by Cushman et al. (2023).

Initial mentoring model and purpose

The mentoring model described here was part of a larger preparation and professional learning program designed to support beginning teachers. The five-year program was created to support individuals with strong content preparation to enter and stay in secondary teaching. The two-way mentoring model was conceived to provide beginning teachers with both a mathematician and a master teacher mentor, starting from their preservice year and continuing during their first four years of teaching (Albert, 2019). In the first year of the program, beginning teachers completed a one-year Master's program and associated student teaching for an initial state credential. Each fellow joined a "high needs" district, as defined by grant funders, in a small or medium sized city. During subsequent years, in addition to mentoring, beginning teachers participated in a professional community of mathematicians and mathematics educators attending monthly meetings and four research colloquia per year. In the monthly seminars, mentors and mentees examined problems of practice in the context of contemporary research. The mathematics education colloquia featured practitioners, educators, and researchers in the field of mathematics or mathematics education. The focus of this study is on the mentoring relationship between the beginning teacher and the mathematician.

For the faculty-teacher mentoring pair, pairs met at least twice per year for a pre-meeting, a classroom observation, and a follow-up meeting. Mentors and mentees were also able to contact each other virtually. Mathematicians were not required to use an observation protocol, but were provided with questions to direct their observations. Additionally, pairs attended monthly seminars and colloquia together.

Methods

The study used qualitative, inductive content approach, guided by Elo and Kyngäs (2008), to extract design principles for a mentoring program between a beginning teacher and faculty mathematician that promoted mathematical *magic*. The data comprised an approximately 60-minute, individual, semi-structured interview with each participant. Participants included seven beginning teachers and three mathematicians. Each teacher had an extensive mathematics content background – six held undergraduate mathematics degrees, and one an engineering degree. Multiple fellows had double majors in subjects including English literature and art. However, the teachers' experiences in schools were more varied, with some who had previously been classroom teachers and others who had not worked in schools professionally. At the time of their interviews, faculty mentors had 10 to 40 years of university experience. Two were current faculty, and one was retired. Two were primarily research mathematicians, while one was primarily a teaching professor.

Because this was a new program, we took an exploratory approach to analyzing the data, using an inductive content analysis, starting with open coding, creating categories and associated memos, and abstracting themes, in this case design principles, from interviews (Elo & Kyngäs, 2008). Further, because the purpose of these analyses was to improve the mentoring between a mathematician and a teacher, special attention was paid to the interactions around mathematics content and pedagogy. Known issues in mentoring were not targeted because while significant, we focused on the unique aspects of this program.

Results

From the analysis, it was apparent that both participating teachers and mathematicians wanted to share with their respective students' ideas of *mathematical magic*. Participants shared a common love for the discipline of mathematics and found mathematics intellectually satisfying. The focus of the results section are the four emergent principles for designing a similar program.

Principle 1: Determine what constitutes mathematical *magic*

All participants described complementary ideas of what constitutes mathematical *magic*. One participating mathematician suggested that mathematical *magic* emerges by using what is known to understand what is unknown. He stated, "mathematics is a lot about pushing boundaries. [...] What happens if we push our boundaries and look for larger and larger structure or different sort of universes to explore?" The mathematician emphasized both a disciplinary principle, examining mathematical structure, and a creative process, problem posing.

A teacher described a complementary conception mathematical *magic* – leveraging patterns to identify generalizations. The teacher stated, "finding ways to capture patterns that we see […] finding ways to create structure out of the abstract ideas." This teacher also provided a disciplinary practice, pattern seeking, and a creative process central to the discipline, generalization.

In undertaking a mentoring program, we suggest that teachers and mathematicians discuss what mathematics is and their own experiences of mathematical *magic*. The focus should be on both the central disciplinary concerns and how those concerns can support creative mathematical endeavors. As they come to a shared definition of mathematical *magic*, mentoring pairs can then proceed to think about lesson designs that might support it.

Principle 2: Apply knowledge of metamathematical reasoning to design for mathematical *magic*

Both participating teachers and mathematicians believed that the teachers' mathematical subject matter knowledge was well developed for their careers as secondary teachers. However, all participants thought mathematicians could bring expertise in metamathematical reasoning to the K-12 classroom. Metamathematical reasoning is an understanding of the organization of the many sub-fields of mathematics, how mathematical knowledge relatively situated, what should be focused on, and how to differentiate between conventions, axioms, theorems, and so on (Dawkins & Roh, 2016). Mentoring pairs felt this metamathematical expertise would be useful to support beginning teachers to a) locate mathematics content in the mathematical universe and b) identify the kinds of skills, habits, procedures, and understandings that would best serve secondary students.

One teacher described that completing a more rigorous version of a high school proof would help her to, "get some better intuition looking at these high school concepts and how can I connect them to something that I might not be thinking of." Thus, mentoring could help beginning teachers understand how to fit a particular mathematical idea into a larger mathematical world, supporting their continuing development of metamathematical reasoning.

Secondly, while teachers bring expertise in the design of their secondary courses, a mathematician might aid a teacher to understand how to organize and locate those concepts within the mathematical universe beyond the existing curriculum. One mathematician thought he could do this be encouraging task design that supports students to ask questions like, "[Is this] something that works in a specific case or works in a specific type of situation, but does it work in a broader class of situations? How?" In this way, the mathematician and teacher could collaborate to identify how the mathematical topic might have implications for additional study.

Finally, metamathematical reasoning is also useful to identify mathematical objects, procedures, or habits that would serve secondary students to achieve a deep level of mathematical understanding. A beginning teacher described the metamathematical guidance that she would hope to gain in the mentoring relationship,

> What are...essential things that you want to instill [in the secondary classrooms] so that later on, you will

> be successful? How would you really think about that in a purely mathematical way, including those mathematical concepts that students need to know or maybe we know how to teach these concepts better?

Specifically, the teacher was seeking mathematically-focused advice around which aspects of an idea should be emphasized and extended.

A focus on metamathematical reasoning could provide additional support for mathematical *magic*. Specifically, a deeper understanding of the mathematical universe might provide secondary students with additional skills to recognize the creative possibilities and intellectual challenges in mathematics. For the mentor pairs, we would suggest programming to discuss and identify metamathematical reasoning.

Principle 3: Sharing knowledge of content and teaching/students across contexts to create opportunities for mathematical *magic*

The mentoring pair described that there were some shared practices that translated between secondary and post-secondary contexts. One mathematician described it as bringing, "the experience of a million years as a math teacher." Based on the data, a mathematician's teaching experience might provide transferable expertise in a) describing and organizing problem-solving approaches, b) identifying and addressing student misconceptions, and c) sequencing and tying content together. According to one mathematician, "[a]n advantage for me is the sense that they know I can't really comment with any depth of knowledge about managing the classroom" or other aspects of the secondary context, but can really focus on the transferable expertise.

However, transferring between contexts can be challenging. One mentoring pair had a conversation connecting curricular content to undergraduate mathematics, but the beginning teacher left the conversation thinking, "I would love to share the information that you're telling me, but how could we break it down [for secondary students]?" On the other hand, another beginning teacher was struggling to help ninth-grade students understand formulas for surface area and volume without using ideas from calculus. After a conversation, the beginning teacher reported that the mathematician-mentor, "was definitely helpful in giving me ways that I can answer those questions to a group of students without the use of calculus." While many mathematics teachers might provide this kind of support, the faculty mathematician's focus on mathematical ideas, rather than the secondary context, provided an opening for this conversation.

To address the challenges of translating expertise across various mathematical contexts, we suggest creating a space for explicit conversations about which student experiences of content are parallel, which are transferable with consideration, and which are not. Through these conversations, mentor pairs can begin to address how to design lessons that best harness their joint expertise and provide fertile ground for mathematical *magic*. Sinclair (2003) suggests that providing diverse perspectives on particular mathematical concepts, these discussions may also encourage real mathematical learning in addition to maintaining teachers' and students' interest in the content. In future iterations, beginning teachers might visit their mathematician-mentors' classrooms to initiate these conversations. Teachers and mathematicians can work alongside each other to design classroom experiences that highlight the *why*, which may lead to innovative experiences for themselves and students across contexts.

Principle 4: Embracing teachers' expertise in students' mathematical development to create mathematical *magic*

Throughout the mentoring experience, mathematicians repeatedly noted that the beginning teachers had expertise in supporting their students and managing the numerous other requirements for teachers. One mathematician noted that in the face of their mentee's classroom challenges, "it makes me kind of question my mantra that the math content is always primary." In addition to the pedagogical expertise, mentors also referenced how teachers' knowledge of adolescent development influences their content decisions. The difference between secondary and post-secondary levels was described as "a different tenor," by one mathematician, saying, "it is also part of connecting to mathematics as a whole. […] I like that it is a vertical viewpoint." Beginning teachers need to understand both their students' psychological development and mathematical learning trajectory.

Given what teachers know about students' development trajectory, a mathematician recognized that this was displayed in how teachers choose to deploy mathematical activities. In the following, a mathematician outlines how the teacher he was mentoring used her knowledge of students to prepare them to learn. The mathematician stated:

> [The students] played an online game and they had to name the type of triangle […] And I thought, you know, this teacher's done no content and just teaches

them terminology. And then I noticed that the students really liked the game, and they really liked that they could name the different types of triangles. [...] I was very surprised at how successful that was, and the last, maybe a third of the class, she's starting to talk about theorems in geometry [...] and the students were ready to listen.

Because university students are primarily adults, it is assumed they will be primed to learn the material as it is being presented. For this mentee, though, it was important for the students to be successful before they were ready to learn. Another mathematician expressed a similar sentiment, pointing out that his mentees needed to "intentionally [engage] individual students by asking questions or by talking to them," which was different from his experience at the university level.

We anticipate that the application of this design principle will support Principle 3. Beginning teachers have expertise in how individual development and engagement mediates the learning of the content. When considering how to design for mathematical *magic*, teachers' knowledge of their students and the resulting developmentally appropriate a sequence are essential. And while mathematical *magic* is a worthy goal, it might be experienced in very different ways by different students, and thus, teachers' expertise will be invaluable in any curricular designs.

Mentoring Program Challenges

In this section, we will highlight the two major challenges that arose in the program. These challenges are separate from the design principles, but still focused on the context of a mathematician-beginning teacher pair. Beginning teachers and mathematicians, by program design, work in different institutions. This results in two challenges that include: a) not being in proximity when challenges arise and b) having allied but not identical professional roles.

The first challenge arose when beginning teachers noted issues, in planning or classrooms, that need to be addressed quickly and cannot always be predicted. The mathematician mentors were not necessarily in the teachers' immediate physical space when such questions arose, due to working at different institutions. Some of this can be addressed by scheduling regular one-on-one meetings and encouraging video conferencing or other virtual mode of communication. We anticipate that this will increase the accessibility and regularity with which teachers can approach mathematicians for support.

The second, perhaps more challenging, issue was that some teachers felt that the mathematicians could not necessarily engage with issues that they faced as beginning teachers. For example, mathematicians did not necessarily have experience with "students specifically two, three, four grade levels behind" their peers. Secondary teaching is done within the context of schools with adolescents, while mathematicians do mathematics in research groups or teach adults. Generally, the mathematicians had little or no exposure to schools, beyond their own or their children's schooling. While each of them had an investment in mathematics education, most of their professional time was not spent in K-12 schools. Both beginning teachers and mathematicians noted that the mentoring experience providing a grounding in the realities of classrooms for mathematicians. However, we suggest that in a revision of this program, we can unpack these different professional experiences and center the conversations in mathematical *magic*.

Limitations of the Current Study

Because our teachers serve populations of students that included racial, linguistic, and SES diversity in districts identified as "high needs," attention to issues of power and equity in mathematics classrooms is required for their work and also need to be addressed in undergraduate and graduate education. During the course of the larger mentoring program, mentoring pairs attended colloquia focused on equitable practices in mathematics teaching. Despite this, the mentoring pairs did not report an explicit focus on equitable practices in the teaching of mathematics. The lack of attention to equity and justice issues in conversations around mentoring may have contributed to the teachers' ideas that mathematicians could not interact about the realities of their schools.

In our full data set, issues of structural inequality and effects on the classroom were mentioned briefly in the mathematicians' observations. This discussion did not extend to help learners navigate their realities or how the various identities of these learners might impact their experiences in mathematics. We suggest continuing to provide additional programmatic reinforcement to support growth in both sets of professionals. Considering equity is an essential part of designing for mathematical *magic*.

Closing Thoughts

The principles outlined here are not the only way that a mathematician and a secondary teacher can contribute to a mentoring relationship through mathematical *magic*. Some of our lingering questions relate to issues that arise from this relationship, and these questions may or may not be related to this goal. However, this kind of mentoring relationship can be useful because it provides some direction for doing something that both teachers and mathematicians, at least those who are participating, wanted to do but were unsure of how to complete.

Additionally, we do believe in the promise of this goal. While the aesthetics of mathematics have previously been identified as frivolous or elitist, they are, in fact, closely linked to areas of student engagement and intrinsic motivation that, when carefully considered, could have a social justice purpose in mathematics education (Sinclair, 2009). Lockhart (2009) argues that students are often required to engage in formalism throughout their K-12 schooling before they are given the opportunities to experience mathematical creativity in post-secondary education. Lockhart has been criticized for not focusing on addressing these issues systematically, rather than individually (Tossavainen, 2014). For these mentoring pairs and a larger program, mathematical *magic* could be supported in the 6-16 academic space, providing a model for a systemic shift.

Additionally, K-12 students have limited conceptions of the work of a mathematician (Latterell & Wilson, 2012). When students do have an opportunity to interact with mathematicians, they came to understand mathematicians' work and expressed more interest in doing this work in their future (Latterell & Wilson, 2012). However, in a previous study with the same participants, we discovered that the mentorship pairing of mathematicians and beginning teachers was a power-free, two-way mutually beneficial connection that was bound by their love of mathematics (Albert, 2019). This aspect embraces the aesthetic nature of mathematical *magic*. In the future, we suggest that mathematicians are given some opportunities to discuss what they learned, regarding the experience of secondary mathematics classrooms, from their visits with their mentees. In addition, beginning teachers could provide opportunities for students to occasionally interact with mathematicians during their classroom visits. This would provide an additional strength for this project.

Finally, and perhaps most importantly, when experiences in designing courses that can support students to experience mathematical creation are combined with engaging students and understanding their contexts, mathematical *magic* can be born. Leveraging the expertise of both teachers and mathematicians, tasks can be designed and implemented with a focus on the aesthetic. Providing this explicit focus will help mathematicians and beginning teachers think through these experiences and their representation at the secondary or early post-secondary mathematical level. These interactions will also allow us to create a reproducible model of designing and implementing these kinds of explorations that embody mathematical *magic*.

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Three Ghanaian Preservice teachers' knowledge and understanding of assessment literacy: Implications for teaching and learning of mathematics

Justice Enu
Kwame Nkrumah University of Science and Technology

Zanele A. Ngcobo
University of KwaZulu-Natal

ABSTRACT This article offers an analysis of preservice mathematics teachers' knowledge and understanding of assessment literacy. We draw from Abell and Siegel's (2011) notion of assessment literacy to explain the salient component of assessment literacy among the preservice teachers who participated in this study. Semi-structured and group interviews were used to generate data from three second-year preservice teachers enrolled in a Diploma in Education program. Data were analyzed using thematic analysis.

While the findings revealed that preservice teachers perceive assessment as an ongoing process aimed at gathering information about students' learning, their knowledge is grounded more on summative assessment. Furthermore, it was noted that preservice teachers view assessment as generic and not specific to mathematics teaching. In terms of conducting assessments, preservice teachers concede that time aligns with purpose and that administering assessments is solely the instructor's responsibility. Based on the findings, we concluded that preservice mathematics teachers have limited knowledge and understanding of assessment literacy. Thus, it is recommended that teacher educators pay particular attention to the evolution of preservice mathematics teachers' conception of assessment literacy. It is also recommended that preservice mathematics teachers' classroom assessment practices be investigated to ascertain how their assessment knowledge informs their classroom practices

KEYWORDS *Preservice teachers, Assessment literacy, Mathematics, Teaching and learning*

Introduction

The advancement of teacher assessment literacy is critical to improving teaching and learning. For example, Wang, Wang, and Huang (2008) posit that teacher assessment literacy is essential to the success of teaching. Of the same view, DeLuca et al. (2013) and White (2009) feel that teachers' assessment literacy enhances the quality of student learning and student learning motivation. Advocating the same point, in 2015, the Government of Ghana launched a program called Transforming Teacher Education and Learning, with technical and financial support from the Government of the United Kingdom. The program aimed to transform the delivery of preservice teacher education in Ghana by improving the quality of teaching at the teacher training colleges. This reform saw the creation of two policy documents: 1) the National Teachers Standard for preservice teachers and 2) the National Teacher Education Curriculum Framework to guide teacher educators, in-service teachers, preservice teachers, and other stakeholders in the education sector. The reform brought about a restructuring of the content and the assessment regime of the teacher training colleges. Some of the major issues relating to assessment contained in these policy documents are as follows:

1) Teacher educators need to prepare preservice teachers to be assessment literate so that they understand and apply the principles and procedures for sound

classroom assessment of learning (summative) and assessment for learning (formative) and

2) Preservice teachers need to know how to use the information from their assessments to support learning (National Council of Assessment and Curriculum (NaCCA), 2018).

The policy calls for preservice teachers to have assessment literacy and to be able to use assessment information to support learning. Therefore, colleges of education need to harness and develop preservice teachers to become assessment literate to ensure they have the knowledge and assessment skills required to be effective when they become in-service teachers. While Ghana's reform policies and framework prioritized the advancement of assessment literacy, a review of the literature has shown that limited research has been carried out on understanding the knowledge of assessment literacy among teacher educators, preservice teachers, and teachers in Ghana, especially in mathematics.

It is no secret that education in Ghana is in crisis (Mereku, 2019) and that this crisis is strongly pronounced in mathematics (Asare & Nti, 2014; Mereku, 2019). Within these parameters, this study aims to understand preservice mathematics teachers' (PsMTs) knowledge of assessment literacy. Focusing on PsMTs is prompted by the fact that the reform program in Ghana emphasizes advancing assessment literacy among preservice teachers. We believe that one's knowledge of assessment informs one's practice in the classroom. As prospective teachers, PsMTs' assessment knowledge will inform their practices when they become in-service teachers. Understanding how they conceptualize assessment literacy would assist in restructuring reforms to advance their knowledge and practice. Therefore, this case study explored the knowledge and understanding of assessment literacy of three second-year Ghanaian PsMTs in the College of Education. To explore this phenomenon, we asked the following research question: How is assessment literacy conceptualized by some preservice mathematics teachers in the [blinded] in Ghana? Conceptualization in this study refers to preservice teachers' understanding of the meaning of assessment, which informs why assessment is to be carried out (purpose of assessment), and the means of doing so (assessment strategies).

Teachers' assessment literacy

Assessment literacy has been defined from different standpoints. Mellati and Khademi (2018) note that assessment literacy is the:

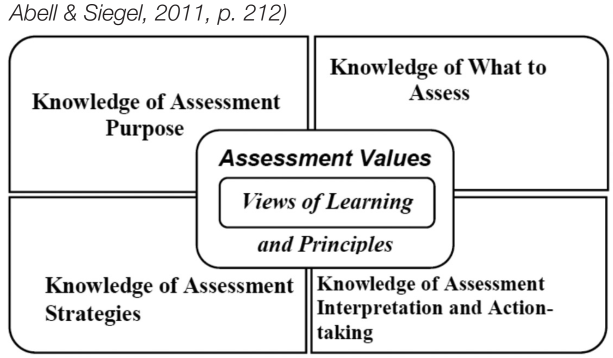
> Possession of knowledge about the basic principles of assessment and evaluation practices, which includes terminology of assessment concepts such as test, measurement, … and the use of assessment methodology and techniques in the classroom … familiarity with an alternative to traditional measurement of learning. (p. 1)

Assessment literacy defines the knowledge and skills one needs to investigate what students know and can do, interpret the results of assessment, and use assessment results to decide how to improve learning (Abell & Siegel, 2011). Popham (2011) contends that "assessment literacy consists of an individual's understanding of the fundamental assessment concepts and procedures deemed likely to influence educational decisions" (p. 265). Stiggins (1999) developed a framework for assessment literacy and proposed seven assessment competencies that teachers need to possess: 1) Connecting assessment to the purpose, 2) Clarifying achievement expectations, 3) Applying proper assessment methods, 4) Developing quality assessment exercises and scoring criteria and sampling appropriately, 5) Avoiding bias in assessment, 6) Communicating effectively about students' achievements, and 7) Using assessment as an instructional intervention. Similarly, Siegel and Wissehr (2011) developed a framework that includes principles, tools, and assessment purposes to explore preservice teachers' assessment literacy. The common aspect among these frameworks is that assessment literacy should advance learning and that knowledge and assessment practices are intertwined. In this study, assessment literacy is defined as the beliefs and knowledge of assessment, based on the framework of Abell and Siegel (2011), because it articulates the key components of assessment literacy, as illustrated in Figure 1.

Abell and Siegel's model considers the various types of assessment knowledge the teacher needs and the

Figure 1

A model for teacher assessment literacy (adapted from Abell & Siegel, 2011, p. 212)



required skills to enhance teaching and learning. In the center of the model are teachers' views of learning, which support the assessment values and principles needed for teaching (Abell & Siegel, 2011). The values and principles interact with four categories of teachers' assessment knowledge: assessment purposes, assessment strategies, assessment interpretation and action-taking, and what to assess. What is emphasized in this framework is the purpose of assessment, meaning that when carrying out an assessment, it is important to consider the purpose for doing so.

The model illustrates the key elements an assessment-literate teacher needs, including a detailed understanding of assessment strategies, which are tools or instruments for gathering evidence of teachers' pedagogy and students' learning. Teachers need to be knowledgeable about formal and informal assessment strategies because they are linked to the type of assessment task that a teacher designs and implements, which in turn reflects their knowledge of assessment purposes and their assessment values (Abell & Siegel, 2011). Knowledge of what to assess refers to teachers' knowledge of mathematics content as concepts and processes. Teachers need to know what, when, how, and why to assess and what to do with assessment data (Abell & Siegel, 2011). These authors explain that a critical component of assessment literacy is what teachers know about interpreting and acting on assessment data. This implies that knowledge of assessment interpretation is about making sense of evidence gathered from assessment, which is born out of teachers' knowledge of the purpose of assessment.

Assessment literacy includes knowing how to use assessment data to help students learn or using evidence from assessments to modify instructional plans. For example, a teacher might use informal questioning as an assessment strategy in a mathematics class when teaching numbers and numerals to learn about students' misconceptions. A teacher might give a formative quiz that asks whether a number with three digits is bigger than a number with two digits, which works in some situations (328 is bigger than 35) but not in the case of numbers with decimals (3.28 is not bigger than 3.5). Based on information generated from the example, the teacher will learn whether another example is needed or if students are ready to move on to a new concept. Teachers' interpretation of assessment as to whether further examples are required or if the teacher needs to modify his/her pedagogy will result from the meaning they make of the assessment data.

Context of the study

This study focused on preservice mathematics teachers in the Central Region of Ghana enrolled in a three-year Diploma in Basic Education program. A total of 18 PsMTs in their second year of study were invited to participate; however, only three (two males and a female) agreed to participate. Data presented here were collected from these three PsMTs. The three PsMTs who participated in this study had been exposed to assessment both as students and students' teachers in micro-teaching (practice teaching that student teachers engage in on campus) and had been exposed to observing teachers teaching and assessing in neighboring schools. An in-depth case study was used to deeply explore PsMTs' assessment knowledge; hence, the three participants were deemed adequate and appropriate. Respondents ages ranged from 17 – 20 years.

Data collection and analysis

Data were collected using individual semi-structured interviews and a group interview. Semi-structured interviews encourage participants to narrate rather than being restricted in terms of what they should discuss. It also allows researchers to probe to understand the studied phenomenon in depth. In this article, the phenomenon being studied was assessment literacy, and therefore, we focus on PsMTs' knowledge and understanding of assessment literacy. Thus, individual and group interviews were deemed appropriate as they allowed PsMTs to narrate their knowledge and understanding of assessment literacy. Moreover, group interviews assisted the researcher in unearthing the dynamics and synergy in PsMTs' knowledge and understanding of assessment. To ensure the trustworthiness of the data collected, the interviews were recorded. This was done to ensure that the real voices and ideas of the participants were captured. Pseudonym IDs were used to protect the identity of the participants (PsMT1, PsMT2, and PsMT3). The numbers allocated have no meaning in terms of performance or order.

To understand PsMTs' knowledge and understanding of assessment literacy and to stay close to the data, the interview data were first transcribed and then segmented (first-level coding) line by line. They were used to generate codes, as shown in Abell and Siegel's (2011) assessment literacy model (Figure 2). The codes can be found in Table 1.

Figure 2

Example of first-level coding of an extract of an interview transcript from one of the participants

How do you understand assessment?

Assessment, as in education, is trying to *determine how best students have followed instruction* **(Evaluating learning)**. So, assessment could be done through quizzes, interviews, and written papers. That is why we have the **formal and informal way of assessing students (Strategies)**

Table 1

Codes generated from participants' verbatim responses

Evaluating learning	Teachers' responsibility	Formal assessment method
Evaluating of learning	Curriculum planners responsible for assessing students	Informal assessment method
Gathering information	Accessing students	Assessment type
Finding information	No specific time Assessment occurs during instruction.	Written-test, interviews Different techniques

In the second analysis phase, a constant comparative method was used, and codes carrying the same meaning were collapsed to identify broader categories, (Glaser & Strauss, 2017). Collapsed codes and broader categories are illustrated in Table 2 below.

Table 2

Collapsed categories to arrive at broader categories

Collapsed codes	Categories	Broader categories aligned to the framework
Gathering information and finding information	Evidence gathering	Knowledge of the purpose of assessment
Evaluating learning; evaluating of learning	Assessing student learning	
Teacher responsibility; assessing students Curriculum planners responsible for assessing students; Assessment occurs during instruction; No specific time	Assessment Implementation	Decision making action taking
Assessment type; Written test, interviews; Different techniques formal assessment method; Informal assessment method	Assessment methods/ techniques	Knowledge of assessment strategies

Findings of the study

To answer the research question about preservice mathematics teachers' conception of assessment literacy, the findings are presented using categories that emerged from the data, which we then align to key constructs drawn from the framework guiding the study.

To ascertain preservice mathematics teachers' knowledge of the purpose of assessment, they were asked to narrate their understanding of assessment:

> **Researcher:** What do you consider to be the purpose of assessment?
>
> **PsMT3:** Gathering information which is educational[ly] important about students so that you can make decisions about them either to intensify the program that they are undergoing or to provide certain remedies to those facing difficulties.
>
> **PsMT2:** Assessment is to find out information about students for decisions and other purposes.
>
> **PsMT1:** To find out how best students have followed the instructions.

PsMT1's views contrasted with those of the other two participants, as this preservice teacher regarded assessment as a means of evaluating students' understanding.

To probe further during the group interview to narrow the discussion to classroom mathematics, the preservice mathematics teachers were asked to narrate their conceptions of assessing mathematics. While all narrated a generic view, PsMT3 went further:

Researcher: What do you consider to be the purpose of assessment, particularly in mathematics teaching?

PsMT3: Mathematics is an integrated subject, so the purpose of assessment is to find out what students know before teaching a topic and after. For example, multiplication builds from addition, so I need to know if they can add whole numbers and use that knowledge to teach multiplication.

PsMT1: I am not sure if it's any different in mathematics; my understanding is that we assess so that we evaluate student learning.

PsMT2: Like in all other subjects, the purpose of assessment is to see if learners have learnt what was taught.

The preservice mathematics teachers conceive assessment as a means to either gather information about the program and students or to evaluate students to ascertain what they know about the topic, thus foregrounding assessment as summative. This positional view contradicts the NaCCA (2018) of Ghana, which emphasizes the importance of both formative and summative assessments. While PsMT3 emphasizes assessing before and after teaching and the need to assess, as alluded to in Abell and Siegel's (2011) framework, none of the preservice mathematics teachers mentioned assessment in learning. This suggests that they do not consider assessment to serve the purpose of ascertaining knowledge and skills while learning occurs.

Preservice mathematics teachers' knowledge and understanding of assessment modes

By modes of assessment, we refer to assessment methods (types or techniques) and timing to administer assessment. Participants shared their views on the forms in which mathematics learning is/can be assessed:

Researcher: What are the modes or techniques through which assessment can be done?

PsMT1: I will say it is only written tests.

PsMT2: We can have tests and anything that will help you to collect information about students.

Thus, PsMT2 added that modes of assessment are any means that offer individuals the opportunity to gather information on student learning. When probed further about other modes of assessment, PsMT2 cited interviews as one of the techniques but was unable to explain how these could be used in assessing learning, suggesting that it is not a method of which she has in-depth knowledge. However, as noted in the above response, PsMT2 emphasized gathering student learning and testing as the common mode of assessment, suggesting that she considers assessment to be summative-driven.

In contrast, PsMT3 stated that assessment could be formal or informal and went on to cite examples of these methods:

PsMT3: … could be done in two forms, either formal or informal. The informal ones are not structured or organized, like quizzes, exercises, and sometimes observations, but the formal assessments are mostly organized and structured, like the end-of-semester examination that we take.

When probing further to articulate the types of assessment adopted in mathematics classrooms, PsMT1 and PsMT2 could not differentiate between types of assessment and techniques used in the mathematics classroom. While PsMT3 mentioned formative and summative assessment, the emphasis was on assessment used for progression purposes:

Researcher: Which mode of assessment and techniques are used in mathematics classrooms? Also, what is the purpose of those modes and techniques in mathematics?

PsMT3: There are two types of assessment, formative and summative, and the techniques used are either tests, assignments, or projects and investigations. Yes, there are other techniques like question and answer, but those are not used for assessing the depth of the content learnt.

PsMT1: As I have said above, assessing is about evaluating students so things like tests and examinations are used to evaluate students.

PsMT2: To gather information about students, we use tests, assignments, examinations, and so on.

It was noted that PsMT3 knows about informal assessment modes but does not believe it enhances learning. While the NaCCA (2018) policy emphasizes assessment for progression, it also emphasizes the importance of informal assessment. However, the preservice mathematics teachers did not mention baseline or diagnostic assessment, suggesting that the conception of assessment is geared towards assessing for progression purposes. Moreover, while knowledgeable about generic forms of assessment, they could not articulate them in relation to mathematics.

Sharing their conception of the timing for administering assessment, they had the following to say:

Researcher: In your understanding, when is assessment carried out?

PsMT3: During the course of study or at the end of the study. So, if it is during the course of study,

> then it becomes a formative assessment. Then, if it is at the end of the course of the study, it becomes a summative assessment. For example, in mathematics, at the end of each lesson, one can administer homework, and at the end of the term, students write examinations.
>
> **PsMT2:** It is carried out from the beginning to the end of instruction …
>
> **PsMT1:** Normally, at the end, but these days, I have to know that the assessment does not have a specific time. We are assessed at any time …

Therefore, according to PsMT3, the time of assessment is aligned with the purpose. PsMT1 and PsMT2 hold the same view and argue that it forms part of teaching and learning. Although PsMT1's conception seems grounded on evaluating learning, the above response shows that new experiences influenced her conception. We based this on her saying that she understands that assessment has no specific time. What is evident in the preservice teachers' responses is that their beliefs influence their conception of timing regarding the purpose of assessment. However, as we noticed in the case of PsMT1, new experiences ignite new conceptions, suggesting that when exposed to new experiences, they are open to changing their conception.

Preservice mathematics teachers' knowledge of decision-making and action-taking in assessment

Narrating their conceptions about decision-makers in assessing, two of the preservice mathematics teachers remarked that assessment is the responsibility of teachers:

> **Researcher:** In your understanding, who are the role players when it comes to assessment or assessing?
>
> **PsMT1:** In my own view, teachers are the key role players because they assess the learners, or those who instruct are the assessors.
>
> Similarly, PsMT3 argued that assessment is done by teachers, further stating that head teachers, as well as the curriculum planning division, can also engage in the assessment process.
>
> While holding a similar view, PsMT2 believes that assessing is the responsibility of teachers and peers:
>
> **PsMT2:** Assessors are those who collect information concerning the students to make value judgments or other decisions. In this case, assessors are teachers, or it can be a student assessing another student.

Although she considers peers as assessors, her response indicates that her conception of assessment is that of learning and places more emphasis on assessors collecting information to make judgments, with limited attention as to how this contributes to learning.

The preservice mathematics teachers' responses show that of the three, only one mentioned peer assessment, suggesting they mostly hold a narrow view of assessment, as they do not consider students as important in the assessment process. All participants echoed that it is the teacher's responsibility because the purpose is to collect information about students or learning. This contradicts scholars such as Bansilal et al. (2011) and Kanjee and Mthembu (2015), who emphasize that assessment is a cyclic process of knowing and informing teachers and learners about the learning process.

Discussion

This study aimed to examine preservice mathematics teachers' knowledge and understanding of assessment literacy. Based on the data from the interviews, preservice mathematics teachers' assessment literacy was categorized into three areas: knowledge of the purpose of assessment, knowledge of modes of assessment, and knowledge of decision makers and action taking in assessment. The standpoint held by the three preservice mathematics teachers reflects assessment as being of learning, contradictory to DeLuca et al.'s (2013) findings in a study conducted in Canada. Their findings revealed that preservice teachers consider building conversations, praxis activities, modeling and critical reflection, and planning of teaching and learning as key components of assessment. This contrasts with the Ghana preservice mathematics teachers in this study, who consider gathering information about students and evaluating students as key assessment components. However, the participants' conceptions in this study coincide with Volante and Fazio's (2007) findings in a study conducted with preservice teachers enrolled in a four-year program in Canada. Their findings revealed that preservice teachers mostly conceive of assessment as summative, meaning that it is mainly considered to be used for evaluation purposes.

The emphasis on assessment being summative was further evident in the conceptions of assessment techniques or strategies held by the participants, as all foreground the use of tests as the main assessment technique. This finding concurs with those of Yilmaz-Tuzun (2008), who conducted a study with 166 preservice teachers in three Midwestern Universities in the US. Yilmaz-Tuzun (2008) argues that preservice mathematics teachers have a limited collection of assessment strategies they choose from. Therefore, they mostly employ questioning and traditional paper and pencil assessment strategies to evaluate students' learning, even though they are aware of other assessment methods. The findings reveal aspects similar to contexts in other parts of the world. Moreover, the findings have educational implications for the Ghanaian education context, which is, in the transformation stage, to ensure the transformation as stipulated in the policy is translated to practice for the development of the preservice mathematics teachers' assessment literacy.

Implications for mathematics classroom teaching

Drawing from the preservice mathematics teachers' views, it could be argued that the generic view of their purpose of assessment is problematic, especially for mathematics teaching. For example, none of the preservice mathematics teachers foreground assessment in learning, while in mathematics classrooms, it is critical that teaching and assessment are constantly intertwined, as posited by Siegler and Wissehr (2011). However, preservice mathematics teachers mostly consider assessment to evaluate learning, i.e., summative assessment. However, summative assessment is important, especially for benchmarking, and since learning is continuous, assessment should also be continuous. Considering that assessment is used to evaluate learning means that feedback only happens after, which does not inform or enhance the learning process. This contradicts the call by Bansilal et al. (2011), which is that feedback should be continuous and aim at informing learning. Mathematics teaching is anchored in both the process and product. Therefore, focusing on one means that holistic teaching is not taking place.

Limitations

This exploratory study involves three preservice mathematics teachers and does not claim to provide findings representative of all preservice mathematics teachers in Ghana. While having only three participants is considered a limitation, the data generated provided an in-depth understanding of the preservice mathematics teachers' conceptualization of assessment. The use of only interviews can be considered a limitation. However, the study focuses on preservice mathematics teachers' conceptions, not practices. Thus, interviews were deemed most suitable to help us answer our research question.

Conclusion

The findings showed that Ghana's preservice mathematics teachers' conception of assessment is that of learning, which focuses on evaluating students and is driven by testing, with teachers having the most important role in the process of assessing. This conception contradicts Ghana's policy documents on assessment, which aim to transform assessment practices in schools and emphasize assessment for learning, assessment in learning, and assessment of learning (NaCCA, 2018).

Based on the findings of this study, we conclude that Ghana's preservice mathematics teachers need exposure to multiple modes of assessment during their teacher training to influence their conception such that it is not one-dimensional but encompasses both formative and summative assessment. As Siegel and Wissehr (2011) mooted, knowledge about a variety of assessment types allows teachers to select the most appropriate and effective instrument to meet their relevant learning needs, exposing preservice mathematics teachers to multiple forms of assessment that are necessary to advance their conception of assessment literacy. As suggested by Kanjee and Mthembu (2015), the development of assessment literacy should be an integral part of teacher training because it influences the quality of teaching in schools.

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NOTES FROM THE FIELD

The Spring 2024 issue features two Notes from the field on the teaching of discrete mathematics. Fran and Vistro-Yu look at the use of "Frankards", a manipulative used in facilitating teaching probability. Pogorelova et al. present a problem-based curriculum in combinatorics to assist in the teaching of the multiplication principle.

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NOTES FROM THE FIELD

Manipulatives in a Mathematics Classroom: The Case of Frankards

Frankie A. Fran
Ateneo de Manila University, Romblon State University

Catherine P. Vistro-Yu
Ateneo de Manila University

KEYWORDS *Frankards, innovation, instructional design, manipulatives*

Introduction

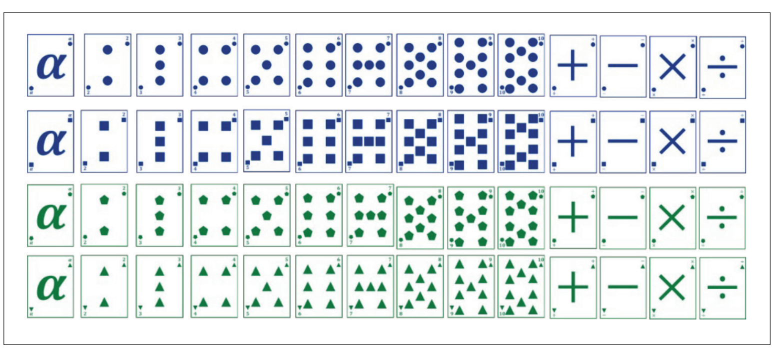
Teachers are tasked with providing opportunities that cultivate and maximize learning. A wide array of approaches can be used to achieve this goal, one of which is manipulatives. We present a teacher-developed manipulative and explain its use in the classroom. We illustrate how this manipulative can be used to teach the concepts of mutually and non-mutually exclusive events.

According to Bartolini and Martignone (2020), mathematical manipulatives are artifacts utilized in teaching mathematics. These are used to explore and investigate mathematical concepts and processes, particularly problem solving. Manipulatives can either be concrete or virtual. The *Encyclopedia of Mathematics Education* defined concrete manipulatives as “physical artifacts that can be concretely handled by students and offer a large and deep set of sensory experiences." In contrast, virtual manipulatives are "digital artifacts that resemble physical objects and can be manipulated in a similar way as their authentic, concrete counterpart" (Bartolini & Martignone, 2020, p. 365). The teacher-developed manipulative, *Frankards*, is “a set of 56 cards specifically crafted to assist teachers in teaching probability concepts through concrete manipulation” (Fran, 2021, p. 2). The deck of *Frankards*, as shown in Figure 1, is a copyrighted work in the National Library of the Philippines with registration number O2021-01.

Typically, teachers use a standard deck of cards to illustrate some ideas of probability. However, they are banned to prevent students from engaging in gambling-related activities. As a result, the first author developed *Frankards* (see Figure 1), which is more mathematical in form than the standard deck of cards. Divided into blue and green cards, *Frankards* is composed of four alpha cards, thirty-six arithmetic cards, and sixteen Multiplication-Division-Addition-Subtraction (MDAS) cards. This was designed to provide students a more engaging learning experience through concrete manipulation as they learn the mathematical concepts.

Figure 1

Deck of Frankards



Note: This figure is adapted from Fran (2021, p. 2).

Teaching and Learning with Manipulatives

Mathematics teachers have long utilized manipulatives, and some have been proven to be effective in teaching some important concepts in mathematics. Students who use manipulatives, specifically algebra tiles, performed better in algebraic operations than students who never worked with the same material (Larbi & Mavis, 2016). Salifu (2022) also suggested that algebra tiles be utilized in teaching algebraic processes such as solving linear equations. Thus, it is recommended that educators use manipulatives in teaching algebra.

Gurung and Chaudhary (2020) showed that learning with concrete manipulatives in a mathematics class results in a more meaningful learning experience for students. Similarly, concrete manipulatives encourage teachers to be more creative in teaching abstract concepts. For manipulatives to be used effectively, learners' real-life experiences must be considered (Eby, 2022). Teachers should also utilize a variety of instructional manipulatives, depending on the topic being discussed. Teachers should take note of how the lessons are designed when using manipulatives. Manipulatives provide an avenue for learners to explore and understand the topics in a meaningful and enjoyable way. This was supported by Furner and Higgins (2022), who pointed out that manipulatives aid and motivate students to learn mathematical concepts and skills.

In cases where students experience anxieties in learning mathematics, Stoehr and Olson (2021) highlighted that teachers' use of manipulatives helps reduce the anxieties through engaging learning activities. With the positive effects of using manipulatives, Kontas (2016) recommended that teachers use these instructional materials more frequently to maximize student learning. It was also noted that manipulatives are instrumental in giving meaning to abstract mathematics concepts.

Aside from helping students create their own representation of abstract mathematical concepts, manipulatives also assist learners in communicating what they have understood to their teachers and peers (Odum, 2022). As with previously studied manipulatives, *Frankards* holds promise of aiding students' comprehension when entering an unfamiliar land, in this case, one involving probabilistic concepts (Fran, 2021).

Using *Frankards* in the Mathematics Classroom

The use of *Frankards* in teaching probability has been shown to contribute to students' achievement with a significant increase in test scores (Fran, 2021). This report will discuss the applications of *Frankards* in teaching probability through illustrative examples. These problems were anchored on the learning competencies set by the Philippines' Department of Education for Grade 10 Mathematics, focusing on mutually and non-mutually exclusive events. Mutually exclusive events are events that cannot occur at the same time. On the other hand, non-mutually exclusive events can happen simultaneously. If the problems involve mutually exclusive events, equation (1) is used. On the other hand, equation (2) is used if the problem involves non-mutually exclusive events.

$$P(A \text{ or } B) = P(A) + P(B) \quad (1)$$

$$P(A \text{ or } B) = P(A) + P(B) - P(A \cap B) \quad (2)$$

Example 1. Mutually Exclusive Events

Let us consider the following problem:

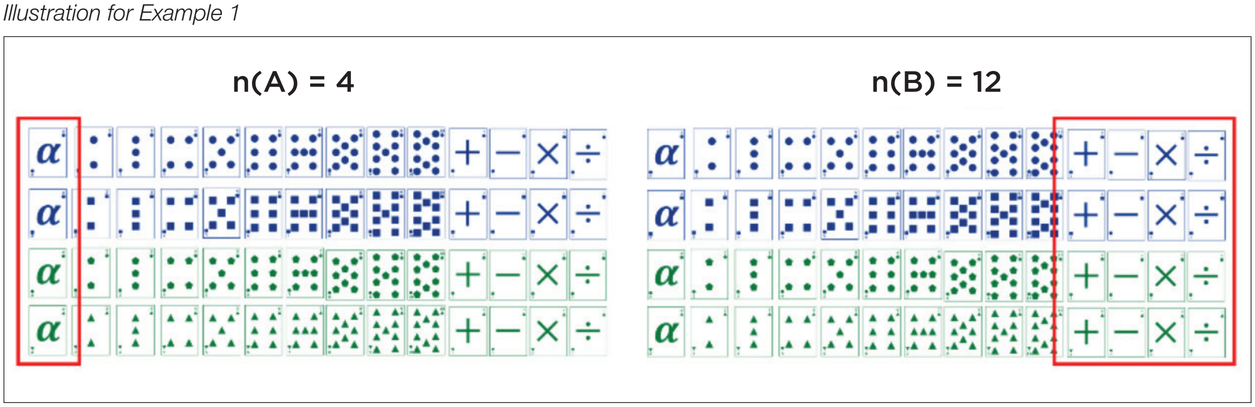
In a standard deck of Frankards, find the probability of getting an alpha or MDAS card.

Solution:

To solve this problem, let A be the alpha cards and B be the MDAS cards. Then, *Frankards* will be utilized, as shown in Figure 2.

Figure 2

Illustration for Example 1



As shown in Figure 2, getting an alpha card and an MDAS card cannot happen simultaneously. Since the problem is a mutually exclusive event, the equation to be used is (1). Hence,

$$P(A \text{ or } B) = P(A) + P(B)$$

$$P(A \text{ or } B) = \frac{4}{56} + \frac{12}{56}$$

$$P(A \text{ or } B) = \frac{16}{56} \text{ or } \frac{2}{7}$$

Example 2. Non-mutually Exclusive Events

Let us consider the following problem:

In a standard deck of Frankards, find the probability of getting a blue or arithmetic card.

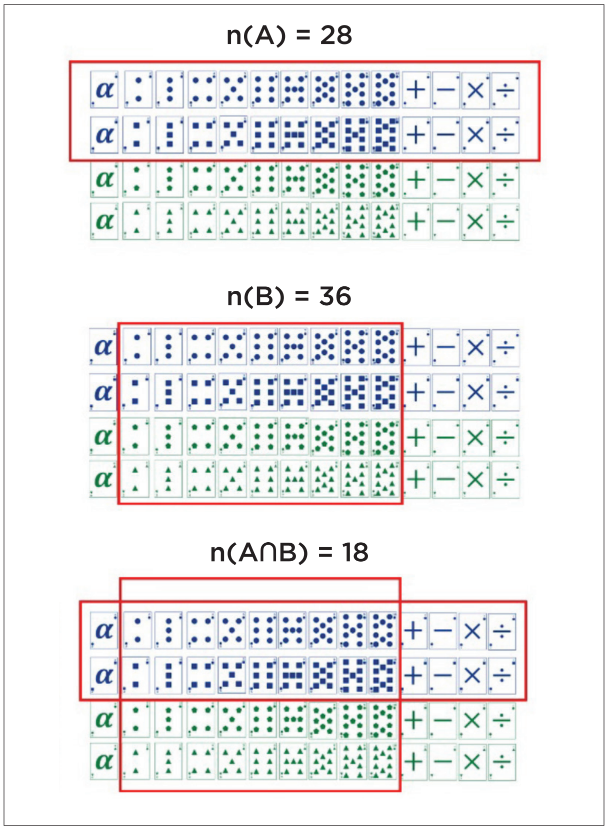
Solution:

To solve this problem, let A be the blue cards, and B be the arithmetic cards. Then, *Frankards* will be utilized, as shown in Figure 3.

As shown in Figure 3, getting a blue card and an arithmetic card shows that the problem is a non-mutually exclusive event. It can be shown that one can draw a blue arithmetic card from the deck. Hence, the equation to be used is (2). Using (2), then,

Figure 3

Illustration for Example 2



$$P(A \text{ or } B) = P(A)+P(B) - P(A \cap B)$$

$$P(A \text{ or } B) = \frac{28}{56} + \frac{36}{56} - \frac{18}{56}$$

$$P(A \text{ or } B) = \frac{46}{56} \text{ or } \frac{23}{28}$$

Feedback from Students and Teachers

The use of *Frankards* in the classroom does not only allow learners to understand the abstract concepts of mathematics. It also engages students in class activities through concrete manipulation (Fran, 2021). As mentioned by one of the students:

"*Frankards* inspired us, students, to love math and to eradicate our math anxiety."

- Student 1

When used in teaching manipulatives, *Frankards* also changes the perspectives of the learners in studying probability concepts. As noted by Student 2:

"Frankards really changed my point of view for cards because who would have thought that playing cards can be used as teaching materials."

- Student 2

Teachers also found *Frankards* to be helpful to them and, more importantly, to the students. As stated by one of the teachers:

"Their [Frankards] use will be more interesting to the students because they look more mathematical in form as the suits that were used are addition, subtraction, multiplication, and division symbols."

- Teacher 1

Generally, both teachers and students found *Frankards* useful in teaching and learning probability.

Concluding Statements

As educators, it is imperative that we provide learners with opportunities to allow learners to reach their full potential. One way is by immersing them in activities that utilize manipulatives. *Frankards*, as discussed in this short report, showcased its potential to be an effective instructional manipulative in teaching probability concepts and mastering numerical skills such as fractions. Hence, it is suggested that *Frankards* be used as instructional material to engage and motivate students in learning probability and other related concepts such as permutation and combination.

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NOTES FROM THE FIELD

A Problem-based Curriculum to Develop the Multiplication Principle for Counting

Lioubov Pogorelova, *New York University*
Stephanie Sheehan-Braine, *New York, NY*
Anthony John, *New York, NY*
Nicholas H. Wasserman, *Teachers College, Columbia University*

KEYWORDS *multiplication principle, problem-based learning, variation theory*

Introduction

Multiplication is relatively simple and is at the core of solving counting exercises. The multiplication principle (MP) specifies the conditions under which we should use multiplication in combinatorics problems. Yet counting exercises are notoriously difficult (e.g., Batanero et al., 1997; Hadar & Hadass, 1981). Given this difficulty, we explore nuances of the MP for counting that address students' difficulties. This paper proposes a problem-based curricular sequence to develop the MP for counting, with a focus on problem-based learning in combinatorics education.

Problem-based Learning and Combinatorics Education

Problem-based learning (PBL) is a pedagogy that is organized around problem-solving activities. PBL facilitates meaning-making, constructing, discovering, and engaging with information with a structured sequence of tasks (Rhem, 1998). Recent studies suggest that PBL is an effective pedagogical approach to mathematics education, increasing students' interest and motivation (Cerezo, 2004; Mahmood & Jacobo, 2019).

Research suggests that students hold misconceptions about multiplication in combinatorics (Lockwood et al., 2015). For this paper, we use Tucker's (2012) definition of the MP (italics added):

> "The Multiplication Principle: Suppose a procedure can be broken into m successive (*ordered*) stages, with r_1 different outcomes in the first stage, r_2 different outcomes in the second stage, …, and r_m *different* outcomes in the mth stage. If the number of outcomes at each stage is *independent* of the choices in previous stages and if the *composite outcomes are all distinct*, then the total procedure has $r_1 \times r_2 \times \ldots \times r_m$ different composite outcomes" (p. 180).

This definition captures the conditions under which multiplication can be used in counting. However, research suggests that students grapple with significant challenges in the context of counting problems (e.g., Lockwood & Purdy, 2019).

The following four collections of PBL exercises highlight the subtleties of the MP in combinatorics by providing pedagogical scaffolds through the structure and sequence of these exercises.

Collection 1

One way we can learn is through contrast, that is, understanding concept x by contrasting it with concept y. Table 1 highlights the fundamental difference between multiplication and addition through contrast of these concepts, albeit in the same context of the exercises.

In order to solve Exercise 1, multiplication must be used in a two-stage process. For example, we might first pick a donut and then a coffee. The resulting breakfasts are composite outcomes—i.e., (donut, coffee) pairs. For each of the five types of donuts, we might choose any of four flavors of coffee, giving us a total of 20 possible breakfasts. In general, with d options for donuts and c options for coffee, $d*c$ (donut, coffee) pairs are possible. Exercise 1 might be used to illustrate the process-oriented approach to counting used by the MP, in which we use ordered stages to perform a procedure that results in composite outcomes.

Table 1

Exercises 1-2

Exercise 1: At your local café are 5 types of donuts and 4 flavors of coffee. You decide to order a breakfast consisting of a donut and a cup of coffee. How many different possible breakfasts could you order?
Answer: 5*4=20.

Exercise 2: At your local café are 5 types of donuts and 4 flavors of coffee. You decide to order a breakfast consisting of exactly one item. How many different possible breakfasts could you order?
Answer: 5+4=9.

Addition must be used to solve Exercise 2. In contrast to the breakfasts in Exercise 1, the breakfasts in Exercise 2 are *not* the composite result of a two-stage process but instead are selections from two disjoint sets—breakfasts consisting of a donut and breakfasts consisting of a coffee. Counting, however, still might take place in two stages. First, we count the five donut breakfasts, and then we count the four coffee breakfasts for a total of 5+4=9 breakfasts. In general, with *d* options for donuts and *c* options for coffee, *d+c* breakfasts are possible. These problems highlight the counting principles to distinguish multiplication from addition while providing the same contextual elements.

Collection 2

A second way we learn is through generalization, a process through which students experience sameness by observing that a concept applies across a range of relevantly similar instances. But this brings up the question of how those instances are similar. Exercises 3-5 in Table 2 introduce several real-world scenarios for counting problems—a café, routes between towns, and a class of students—that can be used to generalize important facets of multiplication in counting exercises.

Compared to Exercise 1, Exercise 3 introduces a new stage to the counting process: coffee with or without cream. Thus, *for each* of the 5*4 breakfasts in Exercise 1, you can now order your coffee in one of two ways—with or without cream. This notion that "for each outcome," there are *x* more options is a critical way to understand the MP. Breakfasts are now (donut, coffee, cream) triples—the result of a three-stage counting process that yields 5*4*2 composite outcomes. Exercise 3 also introduces a dichotomous option (with or without) that is often useful in counting.

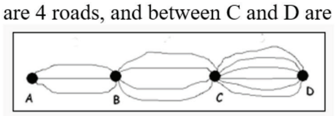
Exercise 4 provides a new context but uses the MP in much the same way as in Exercise 3. Like Exercise 3, part (i) of Exercise 4 requires an approach to counting that uses the MP with three successive ordered stages. This time, though, we have three options in the first stage, four in the second, and six in the third. Hence, the solution is 3*4*6. Parts (ii) and (iii) extend the context, adding even more stages to the process. Part (ii) introduces a modality to the experience in which, *for each* of the 3*4*6 total routes, you could travel in one of three ways—driving, biking, or walking. In part (ii), you cannot switch from one mode of transportation to another—say, driving from town A to town B, biking from town B to town C, and walking from town C to town D. The travel modality must remain consistent for the entire journey. Each composite outcome expands with the modality option, and experiences are now (road A, road B, road C, mode) quadruples. Unlike part (ii), part (iii) allows you to switch from one mode of transportation to another at the towns. For example, if you drive from town A to town B, you can still either drive, bike, or walk from town B to town C. As a result, the distinct composite outcomes change from quadruples to sextuples: experiences are (road A, mode A, road B, mode B, road C, mode

Table 2

Exercises 3-5

Exercise 3: At your local café are 5 types of donuts and 4 flavors of coffee. Coffee is served either with cream or without. You decide to order a breakfast that consists of a donut and a coffee. How many different possible breakfasts could you order?
Answer: 5*4*2.

Exercise 4: Consider towns A, B, C, and D. Between A and B are 3 roads, between B and C are 4 roads, and between C and D are six roads, as shown below.



(i) How many routes are there to get from A to D?
(ii) Suppose you are either going to drive, bike, or walk. How many different experiences are there to get from A to D?
(iii) Suppose on each road you could drive, bike, or walk. How many different experiences are there to get from A to D?
Answers: (i) 3*4*6. (ii) 3*(3*4*6); (iii) (3*3)*(4*3)*(6*3).

Exercise 5: Suppose you have a class of 12 students. Those students are going to have a race, with 1st place receiving a gold medal, 2nd place a silver medal, and 3rd place a bronze medal. How many possible outcomes for the race are there?
Answer: 12*11*10.

C) sextuples. In all of these variants, the actual options in each stage of the counting process are always the same—meaning, no matter which road you take from A to B, you have the same three modality options, and there will be the same four possible roads from B to C.

Exercise 5 explicitly asks the solver to coordinate students with medals and positions. That is, we can consider a solution to be triples of the form (Gold Student A, Silver Student B, Bronze Student C). In this way, there are 12 possible options for the stage identifying which student receives the (first) gold medal, and, subsequently, 11 options for the second silver medal stage, and finally, ten options for the third bronze medal, leading to 12*11*10 possible outcomes. In contrast to the previous exercises, the options in each stage change depending on the choice(s) in the previous stage(s) (i.e., a student who receives the gold medal cannot also be selected for the silver medal).

Table 3

Framework for the Critical Features of the Multiplication Principle

Concept	Critical Feature	Range of Permissible Change
Multiplication Principle for Counting	Order	[We differentiate "outcome vs. process" and "ordinal vs. categorical"] 1. Ordinal outcomes 2. Ordered process and categorical outcomes 3. Unordered outcomes
	Independence	[We differentiate "set vs. cardinality" (i.e., A vs. |A|)] 1. Independent in set and cardinality 2. Independent in cardinality but not set 3. Not independent in cardinality
	Distinct Options	1. Distinct options 2. Non-distinct options (i.e., at least two options are of the same type)
	Distinct Composite Outcomes	1. Distinct composite outcomes 2. Non-distinct composite outcomes

Table 4

Exercises 6-7

Exercise 6:
(i) Suppose you are given the letters ARC. How many different arrangements of all three letters are possible?
(ii) Now, consider from a class of 12 students, there were 12*11*10 possible outcomes for coming in 1st, 2nd, and 3rd place in a race. For each of these two problems, list 5 outcomes. Then describe what it is that makes one outcome different from another one.
Answers: (i) 5 outcomes: ARC, ACR, CRA, CAR, RCA. (ii) 5 outcomes: If we label the students A, B, C, …, L, then: ACD, ADC, LAC, KBL, CDE.

Exercise 7: Suppose you have a class of 12 students.
(i) The class is now going to select a committee of 3 students to be representatives at a larger gathering. How many possible committees of students from the class are there?
(ii) The class is going to select a committee of 3 students to be representatives at a larger gathering. On the committee, there is a President, VP, and Treasurer. How many possible committees of students from the class are there?
Answers: (i) Not 12*11*10. [Answer is 12*11*10/6.] (ii) 12*11*10.

Collection 3

Table 3 highlights four nuances (i.e., critical features) of the MP that correspond to the italicized parts of the definition of the MP mentioned previously: *order*, *independence*, *distinct options*, and *distinct composite outcomes*. The range of possible values for each critical feature is also provided. Each critical feature is accentuated by varying that feature, while keeping the other features invariant.

Order

Although multiplication is commutative, the MP suggests that we adopt an explicit order to the stages in the counting process before we use multiplication (Lockwood et al., 2017; Lockwood & Purdy, 2019). Exercises 6-7 (Table 4) highlight the different ways in which order may appear in counting problems and address the range of permissible change as it relates to order in the MP.

Exercise 6 introduces arrangements. Critical in parts (i) and (ii) is the arrangement of the outcomes themselves (i.e., each letter in the arrangement has an ordinal position). Order also occurs in general, with each element having an ordinal position. For example, we can compare parts (i) and (ii) on the order dimension. ARC is different from CAR, even with the same letters, because the letters are in a different order. Thus, it is not just the subset of letters included but their different orders that the MP enumerates. Use of the MP links to order and includes, as illustrations, exercises having ordinal outcomes—first position, second position, and so forth.

Exercise 7 considers ranges of permissible change of order in the MP. Like part (ii) of Exercise 6, part (i) of Exercise 7 provides a context of a class of 12 students, but the ordered first, second, and third places from the prior exercise have changed to a 3-person committee in which order is irrelevant. The goal is to help students recognize that 12*11*10 is *not* the correct solution—because the use of the MP would include all possible *orderings*. For example, in part (ii) of Exercise 6, ACD was different from DAC, because in the former, A wins the gold, while in the latter, D wins the gold. In contrast, in part (i) of Exercise 7, we could imagine forming a committee in which ACD is identical to DAC, because they are comprised of the same students. For present purposes, students need not know the correct solution—i.e., (12*11*10)/6. The point is to help students identify order as a critical feature of the use of multiplication in counting.

Part (ii) of Exercise 7 asks students to form a committee of three people whose roles are not inherently ordinal and thus differ from the roles in part (ii) of Exercise 6, which are ordinal. Nonetheless, by using an ordered process in selecting the candidates, first for the Treasurer, then the Vice President, and finally the President, the MP treats the (Treasurer, Vice President, President) triples as though they were ordered. That is, we would interpret any of the 12*11*10 triples as ordered: the first selected student is the Treasurer, and so on. Because we draw from the same set of 12 students, the triples (A, B, C) and (C, B, A) would both be included in the 12*11*10 solutions, and would represent different solutions, since in the former, A is the Treasurer, while in the latter, C is the Treasurer. By varying the type of outcomes, this exercise highlights the way in which ordered processes can coincide with outcomes.

Independence

Independence—i.e., the concept that the number of options at each stage of a counting process must be independent of each other—is a critical feature of the MP. Exercises 8-10 in Table 5 highlight the different ways in which independence may inspire the use of the MP.

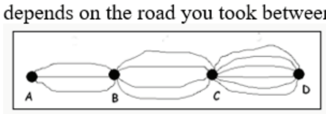
In Exercise 8, the options are independent in both set and cardinality—i.e., to get from B to C, students always choose from the same set of roads, and they always choose from the same number of roads. After listing five outcomes in part (i), students determine in part (ii) whether their choice of road between B and C depends on the road they take between A and B. Using the diagram as a tool, they should recognize that no matter the road they take between A and B, they will *always* have to choose one of the four roads joining B to C—the set of choices they have is independent of any choice of road from A to B. As for part (iii), since the set is the same, its cardinality is too: there are always the same four choices.

Exercise 9 demonstrates a situation that is independent in

Table 5

Exercises 8-10

Exercise 8: Consider the solution to a previous exercise: There were 3*4*6 different routes to get from A to D.
(i) If we label each of the thirteen roads with a unique name (e.g., Abrams Street), give 5 outcomes for routes between A and D.
(ii) Describe whether or not your choice for the road between B and C depends on the road you took between A and B.
(iii) Describe whether or not the number of choices you have for roads between B and C depends on the road you took between A and B.



Answer: (i) Call the streets A, B, C, …, M. Choose from A,B,C first, then D,E,F,G second, then H,I,J,K,L,M last. So: AFM, BEL, BFJ, CDI, CGM. (ii) The choice of D,E,F,G did not depend on the choice for A,B,C. (iii) The number of choices was always 4.

Exercise 9: Each of the following problems has 5*4 as its answer.
(i) Suppose you are given the letters MARCH. Ana is trying to count how many different possible rearrangements of two of the five letters are possible.
(ii) At your local café are 5 types of donuts and 4 flavors of coffee. Suppose you order a breakfast that consists of a donut and a coffee. How many different possible breakfasts could you order?
This means for each of the 5 initial options, there are 4 remaining options. List out the 4 remaining options for each problem. How are those lists different? What do you observe is common between the two problems?
Answers: (i) If M, then A, R, C, H; if A, then M, R, C, H; if R, then M, A, C, H; if C, then M, A, R, H; if H, then M, A, R, C. (ii) C1, C2, C3, C4.

Exercise 10: At your local café are 5 types of donuts and 4 flavors of coffee. Suppose that the café is down to 20 donuts (2 glazed donuts; 4 sprinkled; 1 jelly; 8 cake; and 5 twists), and that you are ordering three donuts, one for Sam, Jenny, and Walter. How many different possible breakfasts for those three friends could you order? Explain why the solution 5*5*5 is incorrect.
Answer: With a limited supply, independence of cardinality does not work (if you choose Glaze, Glaze, then there are no more Glazed donuts and so only 4 options for the third (not 5)). [Actual answer is 111.]

cardinality but not in set. Scenario (b) is analogous to Exercise 1: If the four coffee flavors are {Regular, Vanilla, Hazelnut, Pecan}, then one is always choosing from those same four options, regardless of which donut is selected. Scenario (a), however, is different. If the letter M is first, then the four options for the second letter are {A, R, C, H}, but if the letter A is first, then the four options for the second letter are {M, R, C, H}, and so on. Here, the set of options for the second letter is *dependent* on the choice for the first letter. What is common in scenarios (a) and (b) is that there are always four options—even though they may not be *the same* four options. Thus, independence of the cardinality of the options may come either with the options themselves being independent, as in scenario (b), or with the options changing but always retaining the same quantity, as in scenario (a).

Exercise 10 involves a limited number of each type of donut. Many students may mistake 5*5*5 for the correct solution. The crux of the matter is, however, that the cardinality of options is not independent of prior choices. For example, if you order Sam a glazed and Jenny a glazed, then the café would run out of glazed donuts, and only four donut options would remain for Walter. But if you order Sam a glazed and Jenny a sprinkled, then five donut options would remain for Walter. The number of options changes depending on what happens in prior stages, and we cannot use the MP in this way.

Table 6

Exercises 11-12

Exercise 11: Determine the number of different rearrangements of the four letters of MASS. (i) Explain whether the solution 4*3*2*1 is correct or not. (ii) If we differentiated the S's as S1 and S2, would 4*3*2*1 count the number of rearrangements of the letters MAS_1S_2? *Answer:* (i) Not correct. There are not 4 "distinct" options. (ii) Yes, correct. There are now 4 "distinct" options.
Exercise 12: At your local café are 5 types of donuts and 4 flavors of coffee. Suppose they are down to 20 donuts (2 glazed donuts, 4 sprinkled, 1 jelly, 8 cake, and 5 twists), and that you are ordering three donuts, one for Sam, Jenny, and Walter. How many different possible breakfasts for those three friends could you order? Explain why the solution 20*19*18 is incorrect. *Answer:* 20*19*18 is not correct because the options are not distinct—there are not 20 different donuts—the 2 "glazed" donuts are the same. [Actual answer is 111.]

Table 7

Exercises 13-14

Exercise 13: At your local café are 5 types of donuts and 4 flavors of coffee. You are ordering three donuts, one for Sam, one for Jenny, and one for Walter. How many different possible breakfasts for these three people could you order? *Answer*: 5*5*5.
Exercise 14: Suppose you have a class of 12 students, 8 of whom are American, 2 European, and 2 Asian. The class is going to select a committee of 3 students to be representatives at a larger gathering, with at least 1 Asian and 1 European student on the committee. How many possible committees of students from the class are there? *Answer*: 1*2 + 1*2 + 2*2*12.

Distinct Options

Distinct options are a third critical feature of the MP. Options are distinct when they are distinguishable at each stage of the process. Exercises 11-12 in Table 6 highlight how distinct and non-distinct options influence the application of the MP.

Exercise 11 highlights distinct and non-distinct options using the letters MASS. In part (i), the proposed solution is not correct because MASS does not contain four distinct letters due to the repetition of the letter S. However, in part (ii), the suggested solution would be appropriate for the letters MAS1S2, because the subscripts provide distinct options. Only when the options at each stage are distinct can we use the MP to count.

In Exercise 12, while the café has 20 donuts, the two glazed donuts are the same, the four sprinkled donuts are the same, and so on. For example, suppose we number the sprinkled donuts D3, D4, D5, D6. Then the 20*19*18 solutions would include (D3, D4, D5), (D5, D4, D3), and so forth, which are really all *one* solution: Sam, Jenny, and Walter each get a sprinkled donut. By using a familiar context and keeping all other critical features the same, one can distinguish a situation that contains non-distinct options from a situation that contains distinct options.

Distinct Composite Outcomes

The final critical feature of the MP is distinct composite outcomes. Here, we consider the distinctness not just of the options at each stage in the process, but also of the composite outcomes resulting from completing all stages of the process. From a set-theoretic perspective, this represents whether a one-to-one correspondence exists between the counting process and a set of outcomes representing the solution. See Table 7.

Exercise 13 illustrates the notion of distinct composite outcomes. Even though the donut options are the same for each person, when we consider the composite outcomes, they are distinct. A three-stage process would be to pick a donut for Sam, then Jenny, then Walter, resulting in (Sam's donut, Jenny's donut, Walter's donut) triples. The solution 5*5*5 gives five distinct options for each of the three stages (i.e., the three people). So, a composite outcome might be (glazed, sprinkled, glazed). This is different from another of the composite outcomes (sprinkled, glazed, glazed) because Sam is getting the sprinkled donut instead of Jenny. Thus, the composite outcomes are all distinct from each other.

In Exercise 14, composite outcomes are not all distinct. A seemingly reasonable process would be first to select one of the two Asians to be on the committee, then select one of the two Europeans, and then select one of the ten remaining students (from one Asian, one European, and eight Americans). This process produces a solution of 2*2*10 = 2*2*(1+1+8), which results in overcounting the outcomes because a one-to-one correspondence does not exist between the counting process and the number of outcomes. That is, the composite outcomes are not all distinct. For example, this process would yield both (Asian 1, European 1, Asian 2) and (Asian 2, European 1, Asian 1). However, these composite outcomes are the *same* committee. When the ordered process produces the same solution in more than one way, we cannot use the MP to enumerate the solutions because the composite outcomes would not all be distinct. That is not to say the exercise is unsolvable. Breaking the exercise into three distinct cases, we can use the MP in each of those three cases to obtain a correct count. There are 1*2 committees with two Asian students and 1 European student, 1*2 committees with two European students and 1 Asian student, and 2*2*8 committees with one student of each nationality.

Collection 4

The last collection of exercises highlights the interrelatedness of the four critical features. Exercises 15-17 in Table 8 illustrate how those critical features work together to build a deeper understanding of the MP for counting. We comment on Exercise 15 but leave Exercises 16 and 17 for the reader to contemplate.

The three parts in Exercise 15 draw out three critical features of the MP: independence, order, and distinct composite outcomes. The solution in part (i), 5*4*3 presupposes that you cannot choose two of the same type of donut, and hence that the number of available options at later stages is dependent on the number of available options at earlier stages. The exercise, however, explicitly allows you to order as many of one type of donut as you like. The solutions in parts (i) and (ii) presuppose order and distinct composite outcomes. For example, the 5*5*5 solution in part (ii), which may tempt some students, would generate (glazed, glazed, sprinkled) and (glazed, sprinkled, glazed) as distinct composite outcomes because their order is different. However, they are *not* different breakfasts. Similarly, the 5*4*3 solution in part (i) would generate, for example, (glazed, sprinkled, jelly) and (glazed, jelly, sprinkled) as distinct composite outcomes when, in fact, they are the same breakfast. Thus, the critical features of order and distinct composite outcomes in the MP are sometimes linked. The correct solution in part (iii) requires dividing the exercise into three cases, based on independence—when we have a) three donuts of the same type, b) two donuts of the same type, and c) three different types of donuts. If we have three donuts of the same type, then once we select the first donut—in one of five ways—there is only one option for the remaining donuts. If we have two donuts of the same type, then once we select those two donuts—also in one of five ways—there are four options for the remaining donut. After multiplying 5*4*3 for the third option, division by 6 is needed to ensure that only one of the six possible orderings is being counted. Thus, the cases themselves are constructed to help navigate the issue of the number of outcomes being independent of previous choices so that the MP can be used within each of the three cases.

Table 8

Exercises 15-17

Exercise 15: At your local café are 5 types of donuts and 4 flavors of coffee. You decide to order a breakfast consisting of three donuts, and you can order as many of one type of donut as you like. How many different possible breakfasts could you order? (i) Explain why 5*4*3 is an incorrect answer. (ii) Explain why 5*5*5 is an incorrect answer. (iii) Find the correct answer. *Answer*: (iii) (5*4*3)/6 + 5*4 + 5.
Exercise 16: Suppose you have a class of 12 students. The class is going to select a committee consisting of some number of students to be representatives at a larger gathering. The committee could have 0, 1, 2, 3, … up to 12 students on it. How many possible committees of students from the class are there? *Answer*: 2*2*2*...*2 = 2^12.
Exercise 17: Suppose you are given the letters AEIOUJKLMN. (i) How many different arrangements of 4 of the ten letters are possible? (ii) How many different arrangements of 4 of the ten letters are there that do not begin with A and that end with a vowel? *Answer*: (i) 10*9*8*7. (ii) 8*8*7*4 + 9*8*7*1.

Discussion and Conclusion

This problem-based curriculum was designed as an instructional sequence to develop the MP. More broadly, this sequence of exercises is aligned with variation theory (e.g., Marton, Runesson, & Tsui, 2004). Variation theory describes learning as a process of considering variation against a backdrop of invariance. Specifically, variation theory has four stages: (1) contrast, (2) generalization, (3) separation, and (4) fusion (Marton, Runesson, & Tsui, 2004). The theory suggests that learning a new concept, such as the MP, should begin with contrast, move to generalization, on to separation, and lastly, fusion. The collection of problems presented in this paper aligns with these four stages (i.e., Collection 1 aligns with contrast, Collection 2 aligns with generalization across different contexts, Collection 3 aligns with separation when critical features vary, and Collection 4 fuses those critical features).

The exercises are engineered for a PBL setting. Instructors would assume the role of facilitator as students work independently and collaboratively to construct their own knowledge rather than listening to a traditional lecture (Torp & Sage, 2002). Furthermore, the exercises are meant to set up an explicit, full-class discussion through student presentations and discussion of solutions. This approach will collectively surface the conceptual conclusions about the MP—e.g., using students' solutions to the exercises to elicit the critical features, their subtleties, and their importance for recognizing when and how to apply the MP while solving counting exercises.

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ABOUT THE AUTHORS

Lillie R. Albert, PhD, is a professor of mathematics education at Boston College Lynch School of Education and Human Development. Her research focuses on how Vygotsky's sociocultural historical theory influences mathematical learning across the lifespan, which includes investigating the interaction between content, equity, and leadership practices of mathematics teachers. Collaborating with South Korean scholars, she has recently conducted research on the purpose of mathematics museums and the role that creativity plays in creating cognitively demanding tasks.

Justice Enu has a Ph. D in Mathematics Education. His research interest is in classroom pedagogies and visualization in mathematics.

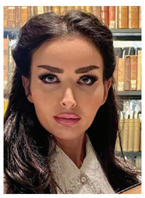
Miriam Gates is a teacher educator who focuses on mathematics and science education across the grade levels. Dr. Gates is particularly interested in the intersection between math or science content, associated disciplinary practices, and considerations for equity. Her current work is focused on classroom approximations of practice for future mathematics and science teachers and how grading practices in higher education can serve to support the professional learning for prospective teachers.

Zanele Ngcobo is an Associate Professor at the University of KwaZulu – Natal, school of Education. Her field of research is in Mathematics Education with a particular interest in Mathematics Teacher Development and Construction of Mathematical Knowledge.

Olajumoke Olayemi Salami, a Postdoctoral researcher in the Department of Science and Technology Education, University of Johannesburg, South Africa. A lecturer at the Federal University Oye Ekiti, Nigeria. Her research interests include curriculum instruction in mathematics education.

Erica Dorethea Spangenberg, a Professor of Mathematics Education in the Department of Science and Technology Education, University of Johannesburg, South Africa. Her research interest includes affective constructs in the teaching and learning mathematics via technology.

Frankie A. Fran is a Ph.D. in Mathematics Education student in the Department of Mathematics, School of Science and Engineering, Ateneo de Manila University, Philippines. He is presently working on his dissertation on ethnomathematics and the inherent mathematical knowledge systems of indigenous communities. He is also an Assistant Professor of Mathematics Education at the College of Education, Romblon State University, Philippines. His research interests include ethnomathematics, instructional design, and culturally relevant mathematics education.



Lioubov Pogorelova teaches courses in business, mathematics, and economics. Inspired by the historical narratives of early explorers, she considers herself a modern-day explorer, navigating uncharted territories through the innovative design of multidisciplinary curricula. Her research interests span calculus and its reforms, ancient celestial navigation techniques, and strategic military tactics. In her pedagogical approach, she emphasizes holistic education that invites intellectual curiosity through studies in mathematics, social sciences, art, and international business.

Catherine P. Vistro-Yu is a Professor in the Department of Mathematics, School of Science and Engineering, Ateneo de Manila University, Philippines. She is the Program Coordinator for Mathematics Education. Catherine was the Philippine representative to the International Commission on Mathematical Instruction from 2008 to 2020. She served in the ICMI Executive Committee as member-at-large for four years. She is currently a member of the IPC for ICMI Study 27 on Mathematics Education and the Socio-Ecological.

Nick Wasserman is Associate Professor of Mathematics Education in the Department of Mathematics, Science and Technology at Teachers College, Columbia University. He teaches mathematics content and mathematics education courses; his work is centered around the mathematical preparation of secondary teachers.

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