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Developing and Supporting Teachers' Mathematical Pedagogy

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AIMS AND SCOPE

The *Journal of Mathematics Education at Teachers College (JMETC)* is a recreation of an earlier publication by the Program in Mathematics Education at Teachers College, Columbia University. As a peer-reviewed, semiannual journal, it is intended to provide dissemination opportunities for writers of practice-based or research contributions to the general field of mathematics education. Although many of the past issues of *JMETC* focused on a theme, the journal accepts articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed.

JMETC readers are educators from pre-kindergarten through twelfth-grade teachers, principals, superintendents, professors of education, and other leaders in education. Articles appearing in the *JMETC* include research reports, commentaries on practice, historical analyses, and responses to issues and recommendations of professional interest.

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PREFACE

The Spring 2023 edition of the *Journal of Mathematics Education at Teachers College* presents three research-based articles that develop and support teachers' mathematical pedagogy. These are then followed by two short reports of contrasting scale: one an implementation of a non-traditional classroom and the other a historical look at a particular mathematics education movement.

Wade et al. open this issue with an investigation into how the Four Component Instructional Design (4C/ID) model can be fitted to support students' development of mathematical concepts. Specifically, using data from the FICSMath Project, the researchers developed a novel framework for supporting the transition from secondary precalculus and calculus to tertiary calculus based on Cognitive Load Theory and the 4C/ID model.

Next, Saclarides takes a more hands-on approach and examines the ways experienced mathematics teacher coaches model instruction for mentee teachers. The researcher performed a case study of one teacher-coach pair, focusing particularly on the interactions between teacher and coach and the scaffolding that occurred during a coaching cycle. The findings noted that verbal asides, written asides in lesson plans, and scaffolding during modeled instruction were important tools for meaningful teacher-coach interactions.

To close this issue, An et al. report on a case study using Mondrian-style art as a vehicle for improving the mathematics pedagogy of pre-service elementary teachers. Pre-service elementary teachers were tasked with creating their own Mondrian-style art pieces and reflecting on the pedagogy used in the activity. Their findings reinforce the utility of visual aids in supporting mathematics instruction while also presenting a unique method in which art and mathematics intersect to improve classroom pedagogy surrounding fractions and ratios.

Mr. Baldwin Mei
Ms. Kaori Yamamoto

Guest Editors

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Presenting a New Model to Support the Secondary-Tertiary Transition to College Calculus: The Secondary Precalculus and Calculus Four Component Instructional Design (SPC 4C/ID) Model

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ABSTRACT Although the secondary-tertiary transition has been investigated in mathematics education research with different focuses and theoretical approaches, it remains a major issue for students in the transition. With success in a science, technology, engineering, or mathematics (STEM) major at stake, we investigated a novel approach to support the transition from secondary precalculus or calculus to tertiary calculus. Using the Four Component Instructional Design (4C/ID) model and empirical data from the United States (US) nationally representative FICSMath project, we mapped instructional experiences of students in the transition to theoretical components of the 4C/ID model. From exploratory factor analysis (n=6,140), we found six factors that mapped to the 4C/ID model components and created the new Secondary Precalculus Calculus (SPC) 4C/ID model. In this model, the Learning Task Component represents tasks to engage learners in meaningful problem solving; the Support Component grounds instruction in reasoning and understanding; the Procedure Component integrates group work and graphing calculators to connect concepts to procedures; and the Part-Task Component represents instruction to develop automaticity. The SPC 4C/ID model presents a unique support for precalculus and calculus teachers in the quest of teaching for learning and transfer of learning across the transition.

KEYWORDS *secondary mathematics, postsecondary mathematics, modeling instruction*

Introduction

Why do we need a(nother) model for teaching precalculus and calculus? Because many in the field—high school teachers and professors alike—have been trying to improve student learning of mathematics to disappointing ends. In the United States (US), for example, students in 12th grade have demonstrated unimproved mathematics scores on the National Assessment of Educational Progress (NAEP) from 2005-2019. Without testable instructional models, we are left with claims about teaching mathematics that are difficult to affirm or reject. We therefore present an empirically based model that focuses on the secondary-tertiary transition in mathematics. Exactly when this transition begins and ends is obscure, yet educational research suggests a period between two years before and after entering university (Gueudet, 2008). In the US, even if secondary students score a 3 or higher (grades range from 1 to 5) on the College Board Advanced Placement (AP) Calculus exams, they may still struggle to perform and persist in college calculus (Atuahene & Russell, 2016). Bressoud (2009) hypothesized that the College Board AP Calculus curriculum is so broad that students move through it learning procedures instead of concepts necessary for success in tertiary calculus. The most common transition support offered is that of bridge courses, e.g., high school-level courses, such as algebra and/or precalculus that are taken (or retaken) at the tertiary level. In the US, tertiary level enrollment in precalculus is well-populated with students who previously completed precalculus in high school. Perhaps surprisingly, retaking precalculus in college does not predict earning higher grades in college calculus (Sonnert & Sadler, 2014). With questionable support from bridge courses and with known variability of preparation quality, a look back to instruction and learning in secondary mathematics is a logical next step.

Theoretical Perspectives

Several theories in the field of mathematics education have been used to investigate the secondary-tertiary transition. Clark and Lovric (2008; 2009) situated the transition in an anthropological framework of the rite of passage. Within their work, they discuss the vast array of changes across teaching styles, the types of mathematics taught, the levels of conceptual understanding, and the advanced mathematical thinking required. Other researchers have investigated the actions, process, objects, and schemata(APOS) theory (Dubinsky & McDonald, 2001; Gueudet, 2008; Selden & Selden, 2001). This theory views mathematical knowledge as being constructed through mental actions that are organized in schemata to make sense of problem-solving situations. Gueudet (2008) discussed the APOS theory relative to students in the transition as they are shifting to more advanced mathematical thinking. While these perspectives have added to our understanding of the transition, the theory foundational to our work is Cognitive Load Theory (CLT).

A major premise of CLT is that working memory load is decreased when domain specific schemata are activated from long term memory. The three sources of working memory load are described as: extraneous cognitive load coming from how material is organized and presented during instruction; intrinsic cognitive load coming from element interactivity, or the interaction of the interconnected parts of the content; and germane cognitive load, which encodes, sends, and connects newly processed information to existing long term memory schemata. A major instructional challenge is how to limit extraneous and intrinsic cognitive load enough so that working memory has the resources to successfully encode information for storage into long term memory. When schemata are built from this process, then learning can occur. CLT defines learning as a *permanent change in long-term memory* (Sweller et al., 1998), and we refer to learning the same way. We also believe that instruction focused on easier-to-present procedures compromises learning (Curry, 2017). This phenomenon is referred to as the *transfer paradox* because such instruction may have an effect on short-term retention for test performance but not on learning (van Merriënboer et al., 2006). This, along with the Four Component Instructional Design (4C/ID) model, created from CLT to support instruction of a complex task (van Merriënboer et al., 2006), is what specifically attracts us to this theory. A complex task, in contrast to simple tasks, has many different solutions, real world applications, cannot be mastered in a single session, and poses a very high load on the learners cognitive system (van Merriënboer et al., 2006). The 4C/ID model was not created specifically for mathematics, however we are applying the model to mathematics because instruction and learning of mathematics is a complex task. For example, mathematics requires multiple solutions during problem-solving (e.g., numeric, algebraic, graphic, etc.), is replete with real world connections, requires time to learn, and—for many—creates a very high working memory load. The 4C/ID model was confirmed for mathematics using data from the Factors Influencing College Success in Mathematics (FICSMath) (Wade et al., 2020). However, there has never been an investigation into how well the theoretical components of the model correspond to the actual instruction of secondary precalculus and calculus teachers. Thus, the purpose of this paper is to explore the fit between the 4C/ID model and senior level high school students' perceptions of how their precalculus and calculus instructional experiences prepared them for college/university calculus.

4C/ID Components

The 4C/ID model components include Learning Task, Support, Procedure and Part-Task Components. These components help understand how to reduce cognitive load and support working memory during the learning of complex tasks. Table 1 presents the model components with their descriptions. Learning tasks ideally connect learners with constituent skills from the support and procedure components that make up the whole task (van Merriënboer et al., 2002). Working with the whole task is challenging yet required for making connections between prior knowledge and new learning. For example, when learning logarithms, the prerequisite concepts of exponents and functions must be used to support learning. Most learners are not cognitively prepared to learn logarithms when there are no schemata developed for exponents and exponential functions.

Figure 1 presents the model as conceived by van Merriënboer et al. (2006). What is important to grasp from the representation of the model is that the Support Component (overarching concepts) is foundational to learning complex tasks. The Procedure Component and the Learning Task Components are established upon the concepts. As presented in Table 1, the partially shaded

Table 1

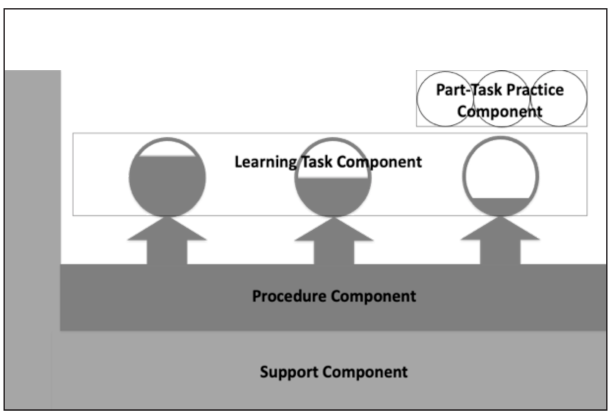
Description of the Theoretical Components of the 4C/ID Model (modified from 4CID.org)

4C/ID Component	Description and Goal of the of the Component
Learning Task Component	Integrates non-routine and routine skills and knowledge with authentic whole task learning experiences, is organized from simple to complex, and provides diminishing scaffolding support (represented by the partially shaded circles).
Support Component	Is foundational to learning tasks as it supports learning of non-routine aspects of learning tasks, explains how to approach problems using cognitive strategies, details how the domain is organized using conceptual models and is always available.
Procedure Component	Specifies how to perform aspects of the tasks through step-by-step instruction, is presented just-in-time and fades as learners acquire more expertise.
Part-Task Practice Component	Provides additional practice for routine aspects to reach a high level of automaticity, provides repetition, begins after routine aspects have been introduced in context of the whole task.

circles represent diminished scaffolding over time of the constituent skills that make up the whole task (van Merriënboer et al., 2002). The Part Task Component represents practice for automaticity, and such practice is known to reduce the working memory load.

Figure 1

Van Merriënboer's Theorized Components of the 4C/ID Model. From Four-Component Instructional Design. *4cid.org.*



Research Questions

The FICSMath project has collected a wealth of empirical data about US students' instruction and learning experiences from their last high school mathematics class before entering tertiary calculus. For this paper, the data analyzed latent constructs to determine how well actual instructional practices, as reported by students in the transition, align with components of the 4C/ID model. Our research questions are thus two-fold: What is the fit of students' perceptions of their instructional experiences, as reported on the FICSMath survey, with the theoretical components of the 4C/ID model? If there is a fit, how can the 4C/ID model be modified to better align with secondary precalculus and calculus instruction?

Data and Methods

The FICSMath Project

The FICSMath project, conducted at the Science Education Department of the Center for Astrophysics | Harvard & Smithsonian remains the most recent US study of high school preparation for college calculus success. Three sources of data were gathered for the development of the FICSMath survey. One source was a broad literature review of current issues in secondary and tertiary mathematics education. Another was a qualitative online survey sent to precalculus and calculus teachers and professors across the nation. Teachers were asked what they were doing, and professors were asked what teachers should be doing, to prepare students for tertiary calculus (for results, see Wade et al., 2016). Lastly, a focus group consisting of experts in secondary and post-secondary mathematics and mathematics education discussed the survey items. Together these provide evidence of content validity. To gauge test-retest reliability, we carried out a separate study in which 174 students from three different colleges took the survey twice, 2 weeks apart. Our analysis found that, for groups of 100, less than a 0.04% chance of reversal between the 50th and 75th percentiles existed (Thorndike, 1997). In the end, the FICSMath survey included 61 questions (many with multiple imbedded items) regarding demographics, course taking, performance levels, and instructional experiences from participants' most recent high school mathematics course before entering single variable college calculus. The survey was administered at the beginning of the Fall

Table 2

The Types of Instructional Questions (14) and Items (70) on the FICSMath Survey

Types of Instructional Questions on the FICSMath Survey	Number of Items	Scale
The amount of conceptual understanding and memorization of procedures required in the class.	2	0-5
The ways calculators were used in class.	7	0-1
Frequency of use of calculators and/or computers in class.	3	0-4
Emphasis on specific types of instruction in class.	7	0-5
Frequency of types of in-class, student-to-student and/or teacher-to-student questioning, responses, and interactions.	10	0-5 and 0-4
Types of problems investigated and solved in class.	9	0-7
How often calculations were checked for reasonable answer.	1	0-5
Types of in-class questions on tests or quizzes.	9	0-1
Use of specific teaching characteristics.	6	0-5
Ways mathematics was connected to real life in class.	4	0-4
Types of support for problem solving given in class.	3	0-4
Types of teaching manipulatives used in class.	3	0-4
Use of in-class assessments.	6	0-4

2009 semester to a stratified random sample of 276 small, medium, and large 2- and 4-year institutions (336 college and university calculus courses or sections). Students completed the surveys in college/university class, and, when the semester was over, the professors reported grades on the surveys before returning them to Harvard University. In the end, we obtained data from 10,437 students from 134 institutions that returned the surveys (73.6% response rate from those who agreed to participate). For the purposes of this study, we included only respondents who had precalculus and/or calculus their senior year in high school and were the next semester in single variable college or university single variable calculus. This reduced the sample to 5,985 respondents. Though the individual percentages of missing values for the variables used were small, multiple imputation was applied to prevent a compound loss of data. Our sample thus included 6,140 cases, a 2.6% increase from the 5,985 respondents under listwise deletion.

Instructional Questions and Items

Table 2 shows the various instructional questions from the FICSMath survey. Instead of choosing which instructional questions should be mapped to components of the 4C/ID model, we included all 70 items (from the 14 questions) in exploratory factor analysis (EFA). This allowed instructional experiences to be mapped empirically instead of theoretically.

Exploratory Factor Analysis

Exploratory factor analysis (EFA) is a widely used and broadly applied statistical technique in the social sciences (Costello & Osborne, 2005). We first investigated the Kaiser-Meyer-Olkin (KMO) statistic, which indicates the proportion of variance in the variables that may be caused by underlying factors. High values close to 1.0 indicate that factor analysis can be usefully applied to the data. The KMO value was 0.848, suggesting that EFA is a reasonable method to investigate the underlying constructs. The large number of participants in the FICSMath study allowed us to meet many of EFA established best practices. For reliable results, the total number of variables in EFA should be at least three to five times larger than the number of expected common factors (Fabrigar et al., 1999). With 70 variables and seven factors, we comfortably met this standard. Additionally, the recommended sample size should have a ratio of 10:1 of observations to factors (Costello & Osborne, 2005) which we also met comfortably with a sample size of 6,140 (after multiple imputation) and seven factors. Table 2 shows that the scales of the variables were different, so we standardized the variables before running EFA. We also selected Maximum Likelihood as the factor analysis method and used eigenvalues greater than 1 (Gorsuch, 1983) and the Scree test to determine the number of factors to keep (Cattell, 1966; Fabrigar et al., 1999). Because, according to Yong and Pearce (2013), larger sample sizes allow smaller loadings, we decided that items with a factor loading of 0.30 or higher would remain in the factors. We also viewed instructional variables as being correlated and hence used the Promax oblique rotational method because of its expedience with larger datasets and the simple structure it can achieve (Gorsuch, 1983; Yong & Pearce, 2013). In the end, 34 out of the 70 variables held together in seven factors. The factors explain 51.9% of the variance within the data. As seen in Table 3, Cronbach's alpha for the factors ranged from 0.618 to 0.873, indicating high internal consistency within each factor.

Table 3

Factors, Constructs, Loadings, and the FICSMath Items (n=6,140; 51.9% of variance explained).

Factors (Cronbach's Alpha)	Latent Constructs	Factor Loadings	FICSMath Items
Factor 1 (0.873)	Ways instruction connects mathematics to the real world and other subject areas.	0.936	Connected math to real life applications.
		0.850	Connected math to everyday life.
		0.705	Examples from everyday world were used.
		0.655	Connected math to other subject areas.
Factor 2 (0.837)	Instruction to support problem solving.	0.928	Teacher highlighted more than one way of solving a problem.
		0.794	Teacher explained ideas clearly.
		0.686	Teacher used graphs, tables, and other illustrations.
		0.599	Teacher presented various methods for solving problems.
Factor 3 (0.746)	Instruction to support mathematical literacy, reasoning, and conceptual understanding.	0.831	Emphasis on precise definitions.
		0.703	Emphasis on vocabulary.
		0.503	Emphasis on mathematical proofs.
		0.421	Emphasis on mathematical reasoning.
		0.412	Emphasis on functions.
		0.331	Extent of conceptual understanding.
Factor 4 (0.747)	Ways calculators were used in the course to support problem solving.	0.626	Allowed to use for trigonometric functions.
		0.614	Allowed to use on exams.
		0.590	Allowed to use to plot graphs of functions.
		0.570	Allowed to use for simple calculations.
		0.566	Allowed to use for homework.
		0.453	Allowed to compute derivatives and integrals.
Factor 5 (0.785)	Frequency of various types of problems solved in the course.	0.822	Frequency of word problems.
		0.706	Frequency of problems with multiple parts.
		0.566	Frequency of problems with written explanations.
		0.507	Frequency of problems with proofs.
		0.485	Frequency of problems being graphed by hand.
Factor 6 (0.712)	Student and teacher classroom interactions to support learning mathematics.	0.941	Classmates taught each other.
		0.775	You taught your classmates.
		0.375	Small group discussions were held.
		0.335	Students spent time doing individual work in class.
Factor 7 (0.618)	Instructional time spent on preparing for assessments and going over assignments.	0.610	Class time spent preparing for quizzes or tests.
		0.580	Time spent reviewing past lessons.
		0.489	Class time spent preparing for standardized tests.
		0.451	Tests or quizzes were given in class.

Table 4

Pearson Correlations Among the Factors (n = 6,140)

	Factor 1	Factor 2	Factor 3	Factor 4	Factor 5	Factor 6
Factor 1						
Factor 2	0.375**					
Factor 3	0.321**	0.454**				
Factor 4	0.231*	0.215*	0.299*			
Factor 5	– 0.031	– 0.006	– 0.071	– 0.067		
Factor 6	0.235*	0.178*	0.115*	0.138*	0.050	
Factor 7	0.260*	0.270*	0.204*	0.254**	– 0.025	0.255*
** Weak correlation (0.00 ≤ r ≤ 0.30); **Moderate correlation (0.31 ≤r ≤ 0.50)*						

Lastly, the Pearson product-moment correlations were computed to determine if the factors held cohesively together among one another. We used Onwuegbuzie and Daniel's (1999) guide of appropriate sample sizes of 800, 84, and 28 to detect small (r = 0.1), moderate (r = 0.3) and large correlation (r = 0.5) levels, respectively. With a sample size of 6,140 we used 0.1 and 0.3 as the cut off for weak and moderate correlations, respectively. As seen in Table 4, all factors were positively correlated with each another except factor 5.

Results

Being comfortable with the factors and how they held together, we determined if and how the factors mapped to the theoretical components of the 4C/ID model. Factors with at least a weak positive correlation (r > 0.1) and theoretical alignment with the components were mapped together. We now present how we perceive the 4C/ID components as they relate to the teaching of high school precalculus and calculus. Table 5 presents the

Table 5

Mapping of the Factors and Constructs to the Theoretical Components of the 4C/ID Model

Factor and Construct from FICSMath Data (correlation value if more than one factor).	4C/ID Component	Description of the Component
Factor 1: Ways instruction connects mathematics to the real world and other subject areas.	Learning Task Component	Integrate new content with necessary prior knowledge and connect with applied problems. Scaffold instruction by integrating the Support and Procedure Components and diminish scaffolding over time.
Factor 2: Instruction to support problem solving. **Factor 3**: Instruction to support mathematical literacy, reasoning, and conceptual understanding (r = 0.454).	Support Component	Support instruction of new learning tasks by explaining how to approach problem solving using cognitive strategies, concepts, reasoning, and mental models (graphs, charts, tables, patterns, etc.).
Factor 4: Ways calculators were used in the course to support problem solving. **Factor 6**: Student and teacher classroom interactions to support learning mathematics (r = 0.138).	Procedure Component	Connect problem solving tasks through step-by-step instruction while integrating prior knowledge with new content. Connection to applied problems create the stage for just-in-time instruction that fades as learners acquire more expertise.
Factor 7: Instructional time spent on preparing for assessments and going over assignments.	Part-Task Component	Provide additional practice through classwork, homework, group work, etc., for problem-solving tasks to reach a high level of automaticity. Repetition begins after new content has been introduced in the context of the whole task.

4C/ID components and addresses our first research question: What is the fit of students' perceptions of their instructional experiences, as reported on the FICSMath survey, with the theoretical components of the 4C/ID model? Factor 1 mapped to the Learning Task Component; Factors 2 and 3 being mapped to the Support Component; Factors 4 and 6 mapped to the Procedure Component; and Factor 7 mapped to the Part-Task Component. Factor 5, a measure of the quantity and variety of problems posed in the course, was negatively correlated and was thus not included in the mapping of factors to model components.

Table 6

Description of the Components of SPC 4C/ID model

SPC 4C/ID Component	Description of the SPC 4C/ID Components
Learning Task Component	• Present learning tasks that integrate new concepts by engaging learners in problem solving that integrates prior learning. • Present applied problems with associated mathematical tasks organized from simple to complex. • Scaffold instruction using the Support and Procedure Components. Decrease scaffolding over time.
Support Component	• Support learning by highlighting various ways of solving problems. • Use mental models to focus on mathematical concepts using graphs, tables, and other illustrations. • Focus on mathematical literacy (definitions and vocabulary), proofs, reasoning, functions, and conceptual understanding to present how the mathematical content holds together. • This component is paramount to building conceptual understanding that will be needed in university mathematics.
Procedure Component	• Use graphing calculators to connect concepts to procedures while being mindful that students will most likely not have access to graphing calculators in college level calculus. • Use small group discussions and group work where students can explain problem solving and where teachers can provide just-in-time guidance.
Part-Task Practice Component	• Provide additional practice to reach high levels of automaticity by reviewing past lessons, going over homework, and preparing for quizzes or tests. • Only use such review after concepts and procedures have been presented to reduce the over reliability on rote mathematical procedures.

The Modified 4C/ID Model

We now address Research Question 2: How can the 4C/ID model be modified to better align with the actual instruction of secondary-tertiary mathematics? Table 6 presents the descriptions of the components of the modified Secondary Precalculus Calculus Four Component Instructional Design (SPC 4C/ID) model. The modified components were generated using Table 3 (factors generated from EFA) and Table 5 (mapping of factors with mathematics education language integrated into the 4C/ID components). The modified SPC 4C/ID model was designed for secondary precalculus and calculus instruction, with the ultimate goal of providing teachers with guidance on ways to better prepare students for tertiary calculus.

Limitations and Future Work

The SPC 4C/ID model was generated from US students' instructional experiences from their senior level precalculus or calculus course, as they reported them on the FICSMath Survey the following semester in single variable college or university calculus. There is no voice, however, representing those who took precalculus and/or calculus in high school and did not proceed to college calculus the following semester. Likewise, this article is silent on instructional practices at the tertiary level—which are also worthy of examination (to see how tertiary calculus has addressed transition issues see Vandenbussche et al., 2018; Norton et al., 2019; & Viera et al., 2019). The SPC 4C/ID model may be unique to a US context, but we hope that mathematics educators and teachers from other countries will consider this model and investigate if it can support students in the transition to tertiary mathematics in their specific institutional structures. Lastly, while the SPC 4C/ID model is derived empirically, it is not yet tested for predictive validity. That is a clear next step for our work.

Discussion and Conclusion

The SPC 4C/ID model was generated by mapping the instructional experiences of students who proceeded to college or university calculus to theoretical components of the 4C/ID model. Initially Wade et al., (2020) used confirmatory factor analysis (CFA) to confirm the 4C/ID model using FICSMath data. In that work, we selected instructional experiences that aligned theoretically with the 4C/ID model. By comparison, this article presents a more empirically robust mapping of the FICSMath data to components of the 4C/ID model because we used no prior assumptions. This allowed all 70 pedagogical items to be used in EFA, rendering 29 items in six factors. The EFA items, correlations of the factors, along with the theoretical implications of the components of the 4C/ID model present an empirically traceable comprehensive instructional approach that aligns with the 4C/ID model. We encourage readers to review Table 6. This is where the SPC 4C/ID model translates to teaching precalculus and calculus. Each of the four l components is accompanied by concrete descriptions of teaching practices (e.g., focus on mathematical literacy using definitions and vocabulary). The basic application of the SPC 4C/ID model is that instruction can be organized around how cognitive load theory (CLT) suggests students learn, with attention to each of the four model components. The SPC 4C/ID model does not require a radical overhaul of what mathematics teachers do—it simply provides an empirically derived model for instruction based on CLT.[1]

Our brief footnote on homework—preserved in the model as one means for Part-Task Practice—illustrates the need to understand that it takes time and practice to learn. Secondary schools whose students cannot reliably find time and space to complete homework must find ways for students to practice.

One key—perhaps *the* key—in providing equitable opportunity are *teachers*. The skills, experience, and effectiveness of mathematics teachers necessarily varies; so too will their ability to teach using a model such as the SPC 4C/ID model. Designing instruction to support learning in precalculus and calculus, and equipping students to transfer that learning into more abstract tertiary calculus, is a complex task. It is also central to the rationale for the coherent model we present.

Mathematics teachers make daily decisions about curriculum, instruction, and assessment often without substantive feedback from students, colleagues, or school administrators. They must find ways, often on their own, to promote student learning. The SPC 4C/ID presents an organized empirical model to support the learning of concepts (Support Component) first and foremost, then other components are built upon the concepts. This model can be used by mathematics teachers to examine and improve their own practices. For example, when teaching trigonometric ratios they may face pressure to make abstract ideas concrete to address the common question of "when will we ever use this?" As a result, they may seek to connect trigonometric ratios to real life applications. The SPC 4C/ID model shows that real life connections must be supported by an understanding of concepts that undergirds the learning of procedures. Rather than answering the "when will we ever use this?" question in isolation, the SPC 4C/ID model suggests that teachers instead seek answers to a more robust question: How can I present students with meaningful problem-solving and real life tasks while providing support for learning overarching mathematical concepts with appropriate attention to procedures and the development of automaticity? It is not a small question, and the answers are not simple, but we hope the SPC 4C/ID model provides a framework that successfully addresses this question.

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[1] One interesting aspect of the model from an application standpoint is the preservation of homework as an item contributing to the Part-Task Practice Component. This is one area of teaching practice where we have seen substantial disagreements among teachers over time (Hansen & Quintero, 2017), especially from an equity standpoint. It is clear to us that, however one feels about homework, the provision of practice was part of the final model. If teachers are unable to assign homework (or students are unable to complete it), students may not get the practice they need. The SPC 4C/ID model suggests that mathematics teachers (and students) still need to find time for practice. This could happen via means other than homework—such as additional class meetings during the year, or calculus courses that take multiple semesters or years to complete—but, as far as we can tell, students need practice. Other components of the model, as outlined in Table 6, may have similar policy or practice implications. We encourage readers to review the model in light of their own contexts.

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Supporting Teacher Learning During Modeled Mathematics Instruction: Findings From One Coach-Teacher Dyad

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ABSTRACT Although modeling instruction has been identified as a productive professional development activity that coaches can use with teachers in their classrooms, coaches are provided with little guidance regarding how to support teacher learning as they model mathematics instruction. While previous research points to the importance of providing teachers with examples of high-quality instruction through the coach's model, teachers may need additional support as instruction unfolds to make sense of what they are observing. As part of the current study, I partnered with one mathematics coach and explored how she explicitly sought to augment teacher learning while modeling mathematics instruction. Findings indicate that the coach leveraged three approaches: engaging the teacher in verbal asides during modeled instruction, providing the teacher with written asides in the scripted lesson plan, and scaffolding the observing teacher's responsibilities. Implications are provided for research and practice.

KEYWORDS *coaching, professional development, modeling mathematics instruction*

Introduction

Given coaching's widespread theoretical and empirical support (Desimone & Pak, 2017; Gibbons & Cobb, 2017; Harbour & Saclarides, 2020), United States school districts are increasingly hiring coaches to support teaching and learning. Here, I use the word "coach" to describe individuals who are tasked primarily with working with teachers on issues related to instructional improvement (Baker et al., 2021). Coaches typically have part- or full-time release from teaching, are stationed at the district office or in schools, and do not evaluate teachers (Campbell & Malkus, 2011). When coaches work with teachers, they may leverage one-on-one coaching activities such as modeling and co-teaching, and group coaching activities such as engaging in lesson study, examining student work, and analyzing classroom video to support teacher learning (Gibbons & Cobb, 2017). Although these individual and group coaching activities are universal (e.g., modeling, co-teaching, lesson study, etc.), coaches select specific coaching activities and tailor their focus to meet their teachers' unique and pressing needs. For example, if a teacher wants to better understand how to implement a teaching through problem solving lesson (Lester & Charles, 2003) with a high-cognitive demand mathematics task (Smith & Stein, 1998) during her mathematics block, the coach might decide to model lessons for the teacher to provide a vision of what this might look like.

This analysis focuses on coaching cycles involving modeling. During modeling coaching cycles, the coach and teacher typically co-plan before instruction; the coach then teaches the modeled lesson in a classroom with students as the teacher observes; and, finally, the coach and teacher jointly reflect about the modeled lesson (Campbell & Griffin, 2017). When coaches model instruction, they provide teachers with opportunities for professional development. Given calls to situate teachers' learning experiences in their own classrooms (Putnam & Borko, 2000) coupled with research stating that the modeler may not understand how to prompt learning for the observing teacher amid modeling (Lunenberg et al., 2007), research is needed that explores how coaches may intentionally support teacher learning as they model instruction in teachers' classrooms.

Previous research has pointed to planning and reflection meetings as offering coaches and teachers rich learning opportunities (Campbell & Griffin, 2017; Russell et al., 2020, Saclarides, 2022a). During such meetings, the coach and teacher have the opportunity to engage in sustained discussions about, for example, student thinking, content, pedagogical dilemmas, and other relevant problems of practice. However, research has yet to delineate the strategies coaches may intentionally leverage to support teacher learning as coaches model instruction for teachers. One line of thought is that, through modeling, coaches expose teachers to high-quality instruction and teachers learn by observing coaches as they enact high-quality instruction to students. In this vein, Lord et al. (2008) stated that the purpose of modeling instruction is to provide "visual images of how standards-based instruction should look" (p. 61). Furthermore, in reference to preservice teachers, Feiman-Nemser (2001) noted that "Teacher candidates must […] form visions of what is possible and desirable in teaching to inspire and guide their professional learning and practice" (p. 1017). Modeling can be viewed as one way to help teachers form this vision of high-quality instruction.

Teachers may need support with processing these representations of high-quality practice and drawing their attention to noteworthy aspects of instruction (Ghousseini & Sleep, 2011). Planning and reflection conversations can provide teachers and coaches with a structured time and place to discuss the modeled lesson, student understanding, and next steps for instruction (Campbell & Griffin, 2017; Saclarides, 2022a). Formal reflection conversations typically take place well after the modeled lesson is over. By then, it may be difficult for the coach and teacher to remember some of the particulars from the modeled lesson. Although informal reflection conversations may take place soon after the lesson is over and while students are still present in the classroom, research has shown that these conversations between the coach and teacher tend to lack depth, which may limit teachers' learning opportunities (Saclarides & Lubienski, 2021). Furthermore, given structural constraints in schools, such as limited time for teachers and coaches to meet during the school day, reflection conversations as a whole are often rushed or do not take place at all (Saclarides & Lubienski, 2021). Hence, how coaches can support teacher learning amid modeled instruction beyond providing teachers with images of high-quality instruction is worthy of empirical investigation. The overarching research question is: how does one mathematics coach support teacher learning during coaching cycles involving modeling?

Methods

Context, Participants, and Case Selection

This qualitative case study (Yin, 2018) took place in a public school district located in a southeastern metropolitan area of the United States, pseudonymously named Southampton. At the time of the study, Southampton enrolled approximately 14,000 students across 11 elementary schools, three middle schools, and three high schools. Southampton sought to provide teachers with high-quality professional development; therefore, the district employed content-focused coaches who engaged teachers in ongoing, job-embedded support in a single academic discipline (e.g., mathematics, English Language Arts, or technology). Southampton coaches did not evaluate teachers, had full-time release from teaching, and worked with teachers in one-on-one and group settings on instructional improvement issues throughout the duration of the school year.

For the current study, I partnered with Beth, an elementary school mathematics coach, and Barbara, an elementary school teacher. All participant and location names are pseudonyms. At the time of the study, Coach Beth was entering her fifth year as a Southampton mathematics coach. Before becoming a coach, Beth taught mathematics in fourth and fifth grades for six and two years, respectively. During interviews that were conducted with Coach Beth at the beginning of this research study to establish context, Coach Beth articulated a vision of ambitious and equitable mathematics instruction that aligns with research-based ideals from the National Council of Teachers of Mathematics (NCTM) (NCTM, 2014) regarding high-quality mathematics instruction. This vision included promoting student-controlled discourse, engaging students in formative assessment strategies to gauge student sensemaking, incorporating high-cognitive demand math tasks into instruction, and promoting conceptual understanding alongside procedural fluency. Entering her first year as a fourth-grade teacher, Teacher Barbara taught mathematics and science only. Teacher Barbara had requested coaching support from Coach Beth in the form of modeled instruction. As is typical during a three-part coaching cycle (Bengo, 2016), Coach Beth typically started her coaching cycle with teachers by engaging them in a planning meeting, followed by either co-teaching, modeling, or observation with feedback, and then closing with a reflection meeting.

The data for this analysis came from a larger study that explored how school-based coaches leverage the one-on-one coaching activities of modeling and co-teaching to support teaching and learning (Saclarides, 2022b; Saclarides & Lubienski, 2021; Saclarides & Munson, 2021). Coach Beth was purposively selected (Yin, 2018) from the larger sample of coaches for the current analysis given that she was able to articulate how she supported teacher learning during modeling.

Data Collection

This analysis rests on three data sources: transcribed participant interviews, transcribed audio recordings of modeled lessons and accompanying field notes, and lesson plans.

I completed a total of four one-on-one, semi-structured interviews with Coach Beth and Teacher Barbara, which were on average 23 minutes long. Coach Beth was interviewed once at the beginning of the study primarily to establish context and understand her emic (Creswell & Poth, 2018), or insider's, perspective on supporting teacher learning during modeled instruction. Coach Beth was also interviewed before and after the modeling coaching cycle with Teacher Barbara to better understand her motivation for modeling instruction for Teacher Barbara and goals for teacher learning, as well as how she sought to support teacher learning while modeling. Teacher Barbara was interviewed after the modeling coaching cycle had ended[1] to understand: what she learned from the modeling episodes, the roles she embodied while Coach Beth modeled instruction, and her interactions with Coach Beth amid modeled instruction.

Additionally, I observed three modeled lessons, which were on average 65 minutes long. These modeled lessons were embedded in one coaching cycle that took place over the course of three consecutive days during Teacher Barbara's mathematics block. During each observation, I generated field notes that attended to verbal and nonverbal coach-teacher interactions. The audio-recordings from the interviews and modeled lessons were transcribed.

Last, Coach Beth shared lesson plans that she had written from a previous coaching cycle with Teacher Barbara. Typically, Coach Beth provided teachers with scripted lesson plans when she modeled instruction. These plans contained three days of scripted lesson plans and all accompanying materials.

Data Analysis

The overarching purpose of this analysis was to better understand Coach Beth's emic perspective (Creswell & Poth, 2018) regarding how she intentionally fostered learning opportunities for teachers during modeling coaching cycles. I began by reading through all interview transcripts from Coach Beth to identify instances where she explicitly discussed how she supported teacher learning during modeling. This analytic reading led to the identification of three approaches: engaging the teacher in verbal asides during modeled instruction, providing the teacher with written asides in the scripted lesson plan, and scaffolding the observing teacher's responsibilities. To triangulate these interview findings (Miles et al., 2020), I examined other data sources (e.g., modeled lesson transcripts, field notes, documents) to potentially uncover additional approaches Coach Beth may have leveraged to support teacher learning amid modeling. Ultimately, I only identified confirming evidence in support of the three approaches she articulated through interviews and did not identify additional approaches.

Last, I used each of these three approaches separately as lenses to understand how Coach Beth leveraged these approaches in practice, as well as how these approaches supported Teacher Barbara's learning. Descriptions of how this analysis was performed for each approach are presented below.

Engaging the Teacher in Verbal Asides

I began by isolating all coach-teacher interactions from the three transcripts of observed modeled instruction, which I define as subsequent turns of talk where the coach and teacher verbally interacted with one another. When describing verbal asides during interviews, Coach Beth stressed that during these kinds of interactions, she briefly paused instruction and made her thinking or reasoning available to the observing teacher, which research points to as being a marker of high-quality discourse to augment teacher learning (Lefstein et al., 2020). For example, this may have included instances in which Coach Beth justified her pedagogical decision making or provided evidence to support claims she made about student sensemaking to the observing teacher. Next, I separated these coach-teacher

1 I had intended to interview Teacher Barbara at the beginning of the modeling coaching cycle as well, but this unfortunately did not happen given scheduling conflicts.

interactions into two categories: those that contained reasoning or evidence to support claims that were made and those that lacked reasoning or evidence to support claims that were made. Interactions that were not categorized as verbal asides included instances in which the coach and teacher interacted about logistical items, such as technology functionality, materials, or student behavior, that were necessary to move the lesson forward but did not seem to promote teacher learning.

After identifying the verbal asides from the observed modeled lesson transcripts, I returned to my field notes, toggling back and forth between my field notes and lesson transcripts, to match up each identified verbal aside from the lesson transcript with confirmatory evidence from my field notes that, indeed, a verbal aside was taking place. My primary purpose in revisiting my field notes was to triangulate findings by method (Denzin, 2001). Last, I revisited Teacher Barbara's interview data to look for evidence of whether and how the verbal asides benefitted her or impacted her learning. Barbara discussed such benefits of the verbal asides in response to interview questions such as "What did you learn from the coaching cycle?" and "Please reflect upon the brief interactions you had with Coach Beth during modeled instruction."

Providing the Teacher with Written Asides in the Scripted Lesson Plan

I started by reading through all lesson plans provided by Coach Beth to identify instances in which Coach Beth included written asides. When describing written asides during interviews, Coach Beth stated that written asides contained reasoning as she sought to give teachers access to her thinking in the lesson plans or details about her anticipation of student sensemaking in the lesson plans. Similar to the verbal asides discussed above, prior research indicates that by making details about thinking or reasoning explicit, this can augment teachers' learning opportunities (Lefstein et al., 2020). Using this definition, I identified eight written asides that Coach Beth embedded into lesson plans. Last, I revisited Teacher Barbara's interview data to uncover whether and how these written asides may have benefitted her or furthered her learning.

Scaffolding the Observing Teacher's Responsibilities

Last, Coach Beth pointed to the importance of carefully scaffolding the observing teacher's responsibilities as the coach modeled instruction, gradually providing the teacher with increasing levels of responsibility throughout the coaching process. Thus, I read through Coach Beth's transcripts to identify the three specific teacher roles she mentioned: sit, observe and take notes; circulate with the coach; and share pedagogical responsibility for enacting instruction. I then used these roles as lenses to re-explore the modeling lesson data, in particular my field notes. By looking for evidence of whether and how Teacher Barbara embodied these roles during modeled instruction, I again sought to triangulate findings by method (Denzin, 2001). Last, I looked for evidence in Teacher Barbara's interview data to understand better how taking on carefully scaffolded roles amid modeling benefitted her or supported her learning. Barbara discussed such benefits of the scaffolded teacher roles in response to the interview question, "Please reflect on some of the roles that you took on as Coach Beth modeled instruction." I also took this as an opportunity to uncover additional teacher roles amid modeling mentioned by Teacher Barbara that perhaps Coach Beth had not mentioned, but did not identify any new roles.

Findings

Engaging the Teacher in Verbal Asides During Modeling

Of the 26 identified coach-teacher modeling interactions, seven (i.e., 27%) were coded as verbal asides as they were marked by reasoning and had the goal of promoting teacher learning. In general, these verbal asides either took place during whole group instruction as the coach paused briefly to engage with the teacher, or during group work time as the coach and teacher jointly circulated. Furthermore, these verbal asides tended to focus on: how the coach and teacher would sequence and select student work samples, the coach's and/or teacher's perceptions of student thinking and understanding, and students who appeared to be struggling. For a fuller description of the kinds of topics that coaches and teachers discuss during modeled instruction utilizing a larger dataset, see Saclarides and Munson (2021).

The following exchange illustrates a verbal aside that took place in the first modeled lesson. Coach Beth and Teacher Barbara discussed the timing for completing their Contexts for Learning unit, which was a curricular resource provided by the school district, with students:

Coach Beth: *So it's like, they're, again, the hard part is that we're actually having to work backwards. So, in hindsight, we probably should have done this in January.*

Teacher Barbara: *Yes.*

Coach Beth: *When they didn't have any-*

Teacher Barbara: *Prior division knowledge.*

Coach Beth: *Well, yeah, just-*

Teacher Barbara: *At least a strategy.*

Coach Beth: *That's right. Like, no use with a strategy.*

In this exchange, Coach Beth and Teacher Barbara discuss how the timing for this unit was not optimal. Importantly, they co-reasoned that because students had already been introduced to the standard algorithm for long division, it was difficult to try to encourage students to use student-invented strategies while accessing the embedded tasks.

To provide a parallel example, during their second modeled lesson, Coach Beth and Teacher Barbara engaged in the following aside in which they discussed their observations of student sensemaking and how to differentiate instruction for two students who appeared to be struggling with a mathematics task:

Coach Beth: *I think that…if I were to do this again I would probably put the two of them [students] together and give them a smaller problem and give them cubes to be able to figure it out.*

Teacher Barbara: *Okay.*

Coach Beth: *Because he's having trouble staying engaged… Josie's keeping him engaged. But yeah I think that probably would be… Josie is…Josie's actually I think has more understanding in the problem and so I don't necessarily think that if I were doing that again I would move her. But, I think those two probably could have used smaller problems.*

Teacher Barbara: *Yeah.*

In this exchange, Coach Beth made her thinking available to Teacher Barbara regarding how to provide further scaffolds and supports for two students who appeared to be struggling. Specifically, Coach Beth would provide the students with manipulatives so they could concretely model the task, and give them a simpler version of the task with smaller numbers.

During an interview, Teacher Barbara reflected on her verbal asides with Coach Beth during modeled instruction:

> There were little conversations about what to do with some of my kids who were a little bit lower academically and just trying to figure out, how do I support them? How do I allow them to succeed in this task without changing it completely and taking it out of context? But, they're still working on a problem using water bottles or using the teacher lounge or whatever the context was. And so, there were several of those conversations…that were just kinda like okay this group is done early, what do I do with them? And so it was a lot of just kind of checking in, seeing if we were on the same page, getting her, racking her brain for advice on different things to do with different students and how to keep them engaged no matter…what their academic ability level was.

In this quote, Teacher Barbara discussed how the verbal asides enabled her to seek out Coach Beth's advice as she modeled instruction. The asides provided opportunities for Teacher Barbara to understand better how to differentiate for students who were struggling, or finished early.

Providing the Teacher with Written Asides in the Scripted Lesson Plan

In addition to engaging Teacher Barbara in verbal asides while modeling instruction, Coach Beth further sought to augment Teacher Barbara's learning opportunities by including written asides in the scripted lesson plans. At the beginning of her modeling coaching cycles, Coach Beth typically provided teachers with scripted lesson plans that detailed her instructional plans with students. Similar to the verbal asides, the written asides were marked by reasoning as Coach Beth sought to give teachers access to her thinking and reasoning or insight about how she anticipated students might engage with the lesson's content.

Across the three days of lesson plans that were analyzed, there were eight instances in which Coach Beth either gave Teacher Barbara access through writing to her reasoning or provided a written narration of her anticipation of student thinking. Instead of being isolated to specific sections of the lesson plan, for example exclusively in the lesson plan closing, these eight verbal asides could be found throughout all sections of Coach Beth's lesson plans. Furthermore, the written asides tended to focus on pedagogy, student thinking and understanding, and mathematics content.

To illustrate, Coach Beth wrote the following in her day one lesson plans:

> I am anticipating making a teacher move here. I think this task is a little bit difficult for the students to complete at this point in the unit, but will provide motivation for getting through the other task of learning about factors and multiples. Therefore, I am anticipating making a teacher move of scaffolding the activity into thinking about multiples. I will give a 100 chart to every student. They will go through the steps of coloring the multiples to see

what generalizations can be made about the multiples of 2 and 3 and possibly 4 if we can get to it.

Coach Beth also wrote an aside to give Teacher Barbara access to her reason for providing students with a particular entry point for solving a math task:

> Tell the students the first place to start thinking might be to go back and re-read the problem. This allows students who do not have an entry point to the task to have something to be thinking about (a question I could ask about something I don't understand) and allows students who are ready to share to have one thing they think is important to share.

During her interview, Teacher Barbara expressed her appreciation for having access to Coach Beth's thinking in this way:

> Getting to hear her predict and anticipate what questions the students were gonna have and how they were gonna respond to different things was really helpful. Because as a first-year teacher I can think of it on my own, and I can try, but at the end of the day I don't have that experience like she has. She's so knowledgeable about all of those things. And so, it was interesting and very, very helpful.

As Teacher Barbara highlights, through teaching children mathematics for eight years and coaching teachers in mathematics for five years, Coach Beth had built up a knowledge base that enabled her to predict how students would engage with content and anticipate potential student misunderstandings. Hence, having access to Coach Beth's thinking seemed to benefit Teacher Barbara as she found this practice "very helpful."

Scaffolding the Observing Teacher's Responsibilities

Last, to support Teacher Barbara's learning during modeling, Coach Beth scaffolded the observing teacher's responsibilities and gradually gave Teacher Barbara more responsibility as the coaching process unfolded. At the beginning of her modeling coaching cycles, Coach Beth expected teachers to sit, observe, and take notes primarily. During an interview, Coach Beth revealed this expectation for teachers during modeling coaching cycles: "She [Teacher Barbara] typically will just sort of observe and take notes. She'll have…[a] checklist and opportunity to just take notes." In reflecting on her early participation in the modeled lessons, Teacher Barbara agreed that she primarily sat, observed instruction, and jotted down notes:

> I sat with a notebook, and I wrote down everything she did and any questions I had or any questions she asked that I thought oh that would be helpful to ask kids moving forward. And so I really just was a learner and trying to soak up just as much information as I could.

This was also substantiated by field notes that were generated during observations.

As the coaching cycle progressed, Coach Beth allowed Teacher Barbara to take on additional responsibilities. For example, she encouraged Teacher Barbara to circulate with her as she monitored students during group work time. During such circulation episodes, Teacher Barbara might interact with Coach Beth by asking her a question or conferring with her about a particular student:

> [A]s the students start working, she'll…just kind of follow me and listen to me talk to the kids. She might ask me why did I ask that question or go, 'I see what you were getting at.' So, it helps me to have the opportunity to make my asides and say, 'Okay, the reason I'm asking this question [is] I want them to see' or 'I want to push them away from this.'

This, too, was substantiated by field notes. For example, during the last 10 minutes of the second observed modeled lesson, teacher Barbara stopped taking notes and instead circulated with Coach Beth given that students were now working in small groups and whole group instruction had ended. The dyad focused their attention on a particular group of students who they perceived needed additional support accessing the task.

After having the opportunity to primarily remain in the learner role by sitting, observing, taking notes, and circulating, Coach Beth started to give Teacher Barbara more shared pedagogical responsibility for enacting instruction. As Coach Beth discussed, "[B]y Day 3 [of the coaching cycle], she'll…take over part of the lesson….Just kind of depending on how comfortable she's feeling with the material and the release and the strategies that kids will use." Thus, as the coaching cycle progressed and Teacher Barbara's confidence grew, she was able to move from observer to lead instructor for several parts of the lesson.

Ultimately, Teacher Barbara appreciated that her participation in modeled instruction was carefully scaffolded. She liked having the opportunity first to be a learner and primarily sit, observe and take notes before taking on more pedagogical responsibility for enacting instruction. Teacher Barbara shared:

> I…like…the way that the coaching was set up…she modeled a day, and then she modeled for the first half [of a day], and then I…followed for the second half, and then I did it by myself. And so, I think the way that that [the modeling coaching cycle] was set up… I'm like…I wanna see it. I wanna hear it. I wanna… really be a part of it. Be able to learn it. So, the way it was set up in that way kind of allowed for me to think through some challenges that have taken place beforehand and kind of see how she handles those.

I now turn to the discussion where I situate this study's findings in the research literature, and provide implications for practice and research.

Discussion and Implications

The overarching purpose of this investigation was to understand better how one mathematics coach supported teacher learning as she modeled mathematics instruction. Coach Beth used three distinct, yet mutually reinforcing approaches to enhance Teacher Barbara's learning opportunities during modeling: engaging the teacher in verbal asides, providing the teacher with written asides in the scripted lesson plan, and scaffolding the observing teacher's responsibilities. Previous research primarily points to the importance of providing teachers with an image of high-quality instruction amid modeling (Feiman-Nemser, 2001; Lord et al., 2008). Hence, this study makes an important contribution to the professional development literature by illuminating other approaches coaches can use to help teachers process these representations of practice and further augment their learning opportunities.

Although all three approaches are related, two of the approaches, engaging the teacher in verbal asides and providing the teacher with written asides in the scripted lesson plans, are more similar in their focus. That is, what unites these two approaches is their focus on making the coach's reasoning available and transparent to the teacher so that the teacher can understand, for example, reasons undergirding the coach's pedagogical decision-making and evidence to support claims about student thinking. Without such access to the coach's reasoning, the teacher may make assumptions that may or may not align with what the coach intended, potentially leading to a missed learning opportunity. Given that previous research on teacher professional discourse found that providing reasoning or evidence for claims made can be generative for teacher learning (Lefstein et al., 2020), this is a practice coaches should consider weaving into their work with teachers to enrich teachers' learning opportunities.

As part of Coach Beth's third approach to support Teacher Barbara's learning, she carefully scaffolded the various roles Teacher Barbara might embody amid modeling. While understandable that a teacher may find it difficult to relinquish control of their classroom, attending to tasks other than observing the coach's teaching may unintentionally shift a teacher's focus and thus limit their learning opportunities during modeled instruction. Hence, it is noteworthy that Coach Beth intentionally scaffolded Teacher Barbara's roles as she modeled instruction, ensuring that the teacher primarily observed and took notes before taking on more roles such as circulating with the coach and sharing pedagogical responsibility for instruction. This aligns with both theoretical (Lave & Wenger, 1991) and empirical (Clarke et al., 2014; Collet, 2015) research which indicates that teacher learning may be augmented when they are provided with a series of carefully scaffolded experiences, or roles in this case, by a more experienced other, such as a coach.

This research has implications for school districts, as well as researchers. Regarding practice-based implications for school districts, coaches may often be told that they should model instruction for teachers. Still, they may not receive proper guidance on leveraging this coaching activity to maximize teacher learning. Hence, it may be beneficial if coaches are provided with high-quality professional development that is ongoing and coherent, involves active learning opportunities, and requires collective participation from coaches as part of a coaching community (Desimone, 2009) to help them understand how to support teacher learning most effectively amid modeling. Such professional development may focus on discussing the three approaches illuminated in the current study, giving coaches time to plan for an upcoming coaching cycle involving modeling and allowing coaches to engage in role play scenarios related to modeling instruction.

Regarding research-based implications, given that the current analysis is based on data that was collected from only one coach-teacher dyad, future research should seek to study the approaches coaches leverage to prompt teacher learning from a larger, more diverse sample of coaches and teachers. Additionally, coaches may need professional development in order to learn how to enact the three approaches detailed in this study to support teacher learning amid modeled instruction. Thus, future research should explore how district-level administrators can most effectively support coaches through professional development as they learn how to enact these coaching practices. Furthermore, given that the current investigation only explored episodes of modeling between the Coach Beth and Teacher Barbara, it is unknown the extent to which the teacher's practice was impacted as a result of working with her coach. Hence, future investigations should consider exploring this important connection—the relationship between coaching and changes in teachers' practice—to better understand the impact of coaching. Last, future research may further unpack the affordances and constraints for teacher learning of each approach discussed in this study.

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Preservice Teachers' Reflections on the Use of Visual Supports to Improve Mathematics Pedagogy: A Case Study of Fraction and Ratio

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ABSTRACT This case study inspected mathematical learning opportunities that employed the results from the participating preservice teachers' (n=75) original visual arts creation process. Data obtained during this study provided empirical evidence for the educational viability of employing original Mondrian-style rectangle sets as a context for generating authentic mathematics inquiry opportunities. The participating preservice teachers generally indicated that through the process of creating their Mondrian-style rectangle sets and exploring the mathematical patterns within their works, such as the presence of fractions and ratios, they were able to develop an improved understanding of the pedagogy for teaching these mathematics concepts. The findings from this study suggest that both mathematics teachers and mathematics teacher educators might better serve their students by including more visual supports when teaching mathematics concepts.

KEYWORDS *mathematics education, teacher preparation, ratios and fractions, proportional reasoning, visual supports for instruction*

Introduction

The integration of visual representations such as tables, charts, symbols, diagrams, and graphs, into mathematics education has been a core pedagogical strategies since ancient times—the Pythagoreans used many of these techniques in their own self-edification and when teaching each other mathematics (Stylianou & Silver, 2004). Visual images used in mathematics can be generally divided into two main categories: (1) illustrative tools for organizing and communicating data; and (2) pedagogical tools for illuminating abstract and theoretical ideas via more tangible representations (Rellensmann et al., 2017). To facilitate the cognition of mathematics, building concept maps using either physical or mental images may help learners to create multifaceted connections between their new and existing knowledge (Friel, Curcio, & Bright, 2001). Visually perceived approaches to mathematics have the potential to facilitate students' mathematical comprehension, while also improving their spatial, proportional and operational execution (An, Cashman, & Tillman, 2021; An, Hachey, & Tillman, 2022).

As a pedagogical approach, visual literacy for mathematical concepts, in other words what occurs when information is illustrated through diagrams and charts, has shown wide applicability throughout multiple mathematical domains, including those embedded within the K-12 curriculum (Ainsworth, 2006). Classroom studies have verified that visual strategies for teaching mathematics generally outperform non-visual alternative strategies, particularly during problem solving tasks that require manipulation of both figures and quantities instead of just one or the other (Boaler et al., 2016). Pedagogically, visual literacy provides an accessible context for providing hands-on opportunities to develop mathematical knowledge, while also expanding the variety of contexts in which teachers can support the occurrence of such learning experiences (Honey et al., 2014; Guyotte et al., 2014).

Learning about mathematical ratios can provide a linkage among arithmetic, algebra and geometry—thereby integrating some of the most challenging instructional components within the elementary mathematics curriculum (Kilpatrick et al., 2001; National Mathematics Advisory Panel, 2008). Unfortunately, due to wide-ranging confusion about the various functions of mathematical ratios, there is a widespread misconception that mathematical ratios are too complex for teaching to early elementary students (Lamon, 2007; 2012). To address this misconception, and to facilitate elementary students' comprehension of mathematical ratios, teachers should offer students abundant opportunities to think proportionally while identifying, analyzing and representing fractional-proportional relationships (Fielding-Wells, Dole, & Makar, 2014). Compared with more commonly-used contexts that are often abstract in nature, such as math problems about speed variations or coupon discounts, visually-perceptible contexts (e.g. chorographical designs and graphic designs) have been used relatively infrequently for classroom illustration purposes (An et al., 2019; Weiland et al., 2021). Therefore, it seems worthwhile to examine potential approaches for developing visual supports into a teaching resource for exploring fractional relationships within elementary mathematics education.

During the past three decades, educational researchers and curriculum theorists have provided scholarship investigating visual supports as a means for improving mathematics education (e.g., Burton et al., 2000; Dana-Picard & Hershkovitz, 2019; Dietiker, 2015; Marshall, 2014; Nutov; 2021; Smithrim & Upitis, 2005). In general, this line of study has confirmed that positive effects on students' academic achievement can be achieved in situations where mathematics learning opportunities are visually supported. Within these studies, many aspects of the connections between the mathematics and visual supports have been articulated and assessed for pedagogical aptitude. However, a discovery from our literature review summarizing this field of study was that empirical research on the topic of mathematics education employing visual supports has as of yet not provided a replicable instructional model for assisting preservice teachers who are in the process of learning to use this pedagogical approach.

Nutov (2021) examined 127 preservice teachers on their learning experiences of learning the concepts of zero and infinity with the integration of visual art works in a mixed method study. The empirical evidence from Nutov's study suggests that mathematics education with visual supports positively correlates with preservice teacher motivation and positive feelings towards mathematics, as well as increased achievement in mathematical problem-solving. Within this context, the goal of this current study was to investigate preservice teachers' conceptual and pedagogical understanding of the relationship between visual supports, such as Mondrian-inspired artwork, and mathematical learning. The specific research question addressed by this research study was: how did the participating preservice elementary teachers evaluate their own mathematics pedagogy that resulted from learning about fractions and ratios with the help of visual supports?

Methods

Participants and Setting

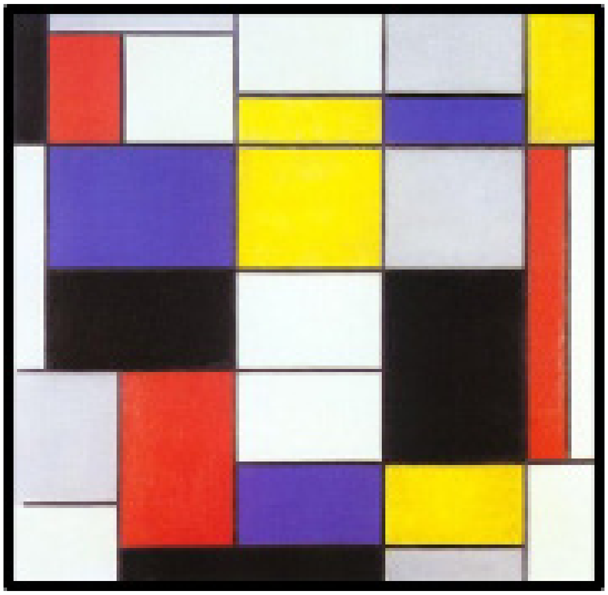
The participants in this case study were recruited from a research university along the southwestern border of the United States. As one of the largest Hispanic serving universities in the U.S., the recruitment location provided a population that was more than three-quarters Hispanic. A total of 75 preservice teachers were recruited from four different sections of an elementary mathematics methods course, all of which were taught by the same instructor and occurred over the duration of two regular academic semesters. The participating preservice teachers were senior-level undergraduates pursuing either an elementary generalist or an elementary bilingual generalist certificate, and a majority of them were working as student teachers in the local school systems. Among the 75 participants, 68 were female and 7 were male. Also, 71 of them self-identified as Hispanic and 4 of them self-identified as Caucasian.

Target Task and Procedure

During this study, the target task had three major components: (1) create an original piece of artwork consisting of a number of shaded shapes drawn in such a way to create a larger shape; (2) identify fractional values within the artwork for each shaded shape—each shaded area serving as a numerator and the whole area as the denominator; and (3) analyze the ratios of those fractional values—the ratio between the length and width

Figure 1

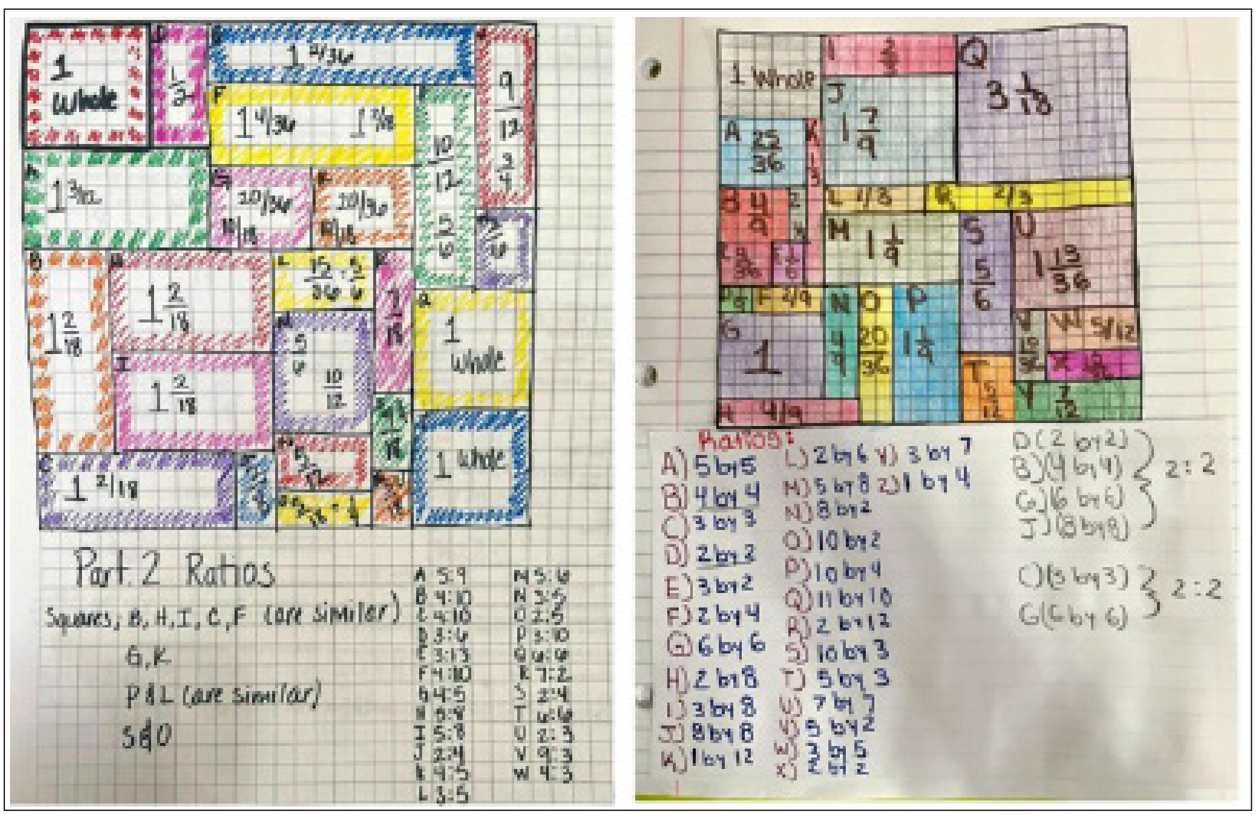
A painting titled "Composition A" (originally created in 1923, photo is public domain) by Piet Mondrian showing his highly geometrical style of art



within each shaded area. In the first component, all participants were taught about the painter Piet Mondrian (1872-1944) and the highly geometrical style employed in many pieces of his artworks (see Figure 1), after which they were guided through the process for creating their own original Mondrian-style rectangle sets on a sheet of letter-sized grid paper (see Figure 2). The definition of a single unit (i.e. *one whole* in fraction) was set as a 6×6 matrix with 36 tiny squares premarked on the top-upper corner of each sheet of grid paper that was distributed to the participants. To ensure sufficient complexity within the resulting artwork created by the participants so that they would have sufficient mathematical relationships to explore, the task directions required each participant to make at least five horizontal and five vertical lines in their artwork, thereby generating at least 25 rectangles. Next, participants were asked to determine the fractional value of each rectangle within their artwork by comparing each rectangle with the size of *one whole* unit as defined in the directions. Lastly, participants were asked to identify all

Figure 2

Example of a Mondrian-style rectangle set with corresponding fraction and ratio information created by two preservice teachers



of the geometrically similar rectangles, meaning those with equivalent ratios, that were present within their artwork.

Next, participants were tasked with identifying the correct fractional values of the rectangles within their original Mondrian-style designs. The predefined 6×6 square provided on the grid paper was used as a key reference by the participants as they computed each rectangle's fraction. The most commonly employed strategy was to construct a fraction by using the total number of grid squares as the numerator and 36 as the denominator. By using this method, rectangles that looked different but had the same areas nonetheless produced equivalent fractions. The emergence of mixed and improper fractions within the original Mondrian-style rectangle sets was an anticipated result from the first target task, because of the limited size of the provided grid paper. The emergence of equivalent ratios (based on part-to-part relationships between length and width) among each pair of geometrically similar rectangles from students' own artwork was an anticipated result from the second target task. The researchers observed that whenever the participants developed a rectangle that was larger than 36 grid squares, a fraction with a value larger than one whole would be formed. It should also be noted that while the participants each created their own original artworks based on the specified criteria provided in the task directions, they were also following a documentation process describing their experience and results. To demonstrate an understanding of fractions, the participants were directed to write down the fractional values presented within each rectangle on their artwork. To demonstrate understanding of ratios, the participants were asked to identify the ratio between width and length for each rectangle, and then write down all similar rectangles with equivalent length-to-width ratios on a separate piece of paper.

Data Collection and Analysis

Data collected in this case study included written essays, pictorial data in the form of original artwork, and hand-written documents for showing fraction and ratio attached to their artworks. To assist the data collection process, two online discussion forums were set up for participants so that they could upload their documents, as well as review each other's creations. The time span for the data collection was four days, which allowed each participant enough time to review the other's participants' artworks and their documented mathematical thinking. Specifically, first an online forum was created for participants to exhibit their original creations as well as share their corresponding mathematical analyses. Then, another online forum was provided three days later after the debriefing where each participant made additional comments regarding their own and each other's artwork and the researcher highlighted the two typical error patterns from existing art explorations (1) treating ratios as fractional values, and (2) employing ratios as if they were additive. In the first type of error pattern, some participants mistakenly treated the fractional values of each individual rectangle's area as interchangeable with the width to length ratios of similar rectangles. In the second type of error pattern, some participants have misconceptions that a rectangle with a width to length ratio of 2:3 has an identical ratio to a rectangle with a width to length ratio of 3:4 because the case is increased by 1 unit which is a "smooth" proportional change. This second forum was used by participants to share post-task reflection essays describing their thoughts on this pedagogical approach for teaching mathematics to elementary students.

The collected data were analyzed utilizing a grounded theory approach (Corbin & Strauss, 2008). Specifically, the collected pictorial data and corresponding written data were paired up and analyzed together as a means for investigating any potential linkages between the participants' original artwork designs, mathematical problem-solving strategies, and the contents of their pedagogical reflections. A content analysis was undertaken with the primary goal of pinpointing and describing the distinguishing methods, strategies, and judgments performed across the entire set of data from the 75 participants. During the first round of coding, individual emergent themes from the collected data were clarified with the aim of generating a broad-spectrum understanding of the participant's general reasoning processes along with identifying any misconceptions about fractions and ratios. This task was accomplished through constant comparisons until the initial entry categories were qualitatively saturated. During the second phase, the data were refined by escalating quotations worthy of being reported, including those pertaining to interesting problem-solving strategies and noteworthy pedagogical reflections; this phase also included sharpening the refinement of the selected representational cases while also eliminating repeated entries with similar content. To minimize any unconscious coding biases, two separate coders worked independently throughout the data analysis process and the results were collectively compared to determine confirmation, or the lack thereof, for the results that emerged. The inter-rater agreement rate was about 93% and the 7% emergent inconsistencies in the coding results were then resolved through open discussion conducted among the research team members until a consensus was achieved.

Results

The participating preservice teachers collectively indicated that by creating their own Mondrian-style rectangular shape sets, they were guided to explore the mathematical patterns such as fraction and ratio within their produced works. They were able to develop an improved understanding of fractions and ratios, as well as the pedagogy appropriate for teaching these two mathematics concepts. A total of 14 themes emerged (see Table 1) after analyzing the collected data and these themes were classified into four major categories, which included: (1) participant identified math content knowledge challenges (e.g., clarified the differences between fraction and ratio); (2) participant developed deepened conceptual understanding of specific math topics (e.g., illustrated parts-to-whole relationship); (3) participant identified the pedagogical connections among math topics (e.g., taught fractions with connections to ratios); and (4) participant generated a positive learning environment (e.g., improved engagement in mathematical investigations). To illustrate the pedagogical reflections in detail, seven participants' testimonies—Evelyn, Joanna, Sabrina, Paula, and Yasmin (the names of the selected participants have been changed to pseudonyms)—were selected to emphasize their distinctive perspectives.

By manipulating a set of rectangles with different length and width characteristics, the participants were provided an opportunity to constructively identify the patterns displayed between the emergent fractions and corresponding ratios. This process appears to have enhanced the participants' self-reported knowledge for teaching mathematics. As an illustration, Evelyn shared personal insights into her own challenges understanding the conceptual differences between fractions and ratios:

> After completing this activity, I realized out of all my educational years, even now that I am all ready to finish my degree in university, I never really worked with ratios with a hands-on activity before and was never taught the difference between a ratio and a fraction. I found that this interactive visual activity helped me reason and be able to picture each rectangle and the relation it had to the whole. Although

Table 1

Themes from Participants' Reflections on Creating Mondrian-Style Rectangle Sets

Major Theme	Specific Theme	Count / Rate (n=75)	
Identified the math content knowledge challenges	Identify flawed understanding of proportions	42	56.0%
	Clarify the differences between fractions and ratios	31	41.3%
	Differentiate additive and multiplicative relationships	7	9.3%
Deepened conceptual understanding of specific math topics	Conceptualize parts-to-whole relationship	39	52.0%
	Conceptualize equivalent fractions concept	31	41.3%
	Conceptualize parts-to-parts relationship	25	33.3%
	Conceptualize improper fractions concept	14	18.7%
Identified the pedagogical connections among math topics	Teach fractions with connections to ratios	70	93.3%
	Teach ratios with connections to geometry/measurement	29	38.7%
	Teach fractions with connections to geometry/measurement	23	30.7%
	Teach ratios with connections to algebras	4	5.3%
Created positive learning environments	Improve engagement in mathematical investigations	34	45.3%
	Embed math knowledge within fun tasks	28	37.3%
	Flexibly adjust difficulty for different grade-levels	11	14.7%

> both fraction and ratio have the concept of part and whole, but ratio is not restricted to the part-to-whole relationship all the time, it may display a part-to-part relationship (e.g., length vs width) as we explored in similar rectangles.

Likewise, another participant named Joanna shared her insights about the pedagogical potential from creating original Mondrian-style artwork as a method for learning about differences between fractions and ratios. Joanna stated that drawing multiple rectangles simultaneously within her original Mondrian-style artwork allowed her to see the ratios emerging among the different rectangles from an empirical perspective. For her, the most challenging part of the ratios-identification task was the rule against making any conclusions based merely on appearance, as she found the requirement of providing mathematical evidence to be quite an unfamiliar approach compared to her traditional, visual method for addressing these types of problems. Joanna explained this peculiar scenario in her own words:

> Even though I consider myself to be pretty good with fractions, I was stunned by observing so many assorted rectangles in this artwork. I was able to see dozens of fractional relationships across all rectangles because they came directly from a whole picture that I could constantly compare and that engaged me with a clear perspective of what I was working. In a ratio sense, I think it is difficult because our eyes can be very deceiving. Especially for the rectangles that are slightly longer or shorter in their length/width, it was hard to tell whether their ratios are equivalent or not because our eyes can't catch these nuances. I need to mathematically analyze the rectangles using the [grid] squares inside and relate them to each other by determining an even scale at which the rectangle either proportionally increases or decreases.

Many participants stated that Mondrian-style rectangle sets offered opportunities to pedagogically connect mathematics topics, especially between number and geometry. Sabrina described her reflections about the pedagogical value of the visual supports employed, and how during her experience the Mondrian-style rectangle sets empowered her to uncover methods for explaining the mechanics of how fraction works, while also linking concepts about fractions with concepts pertaining to geometry and proportion. Sabrina remarked:

> When the concepts are presented separately, fractions are commonly associated only with numerical equations. Now when using Mondrian designs, teaching fractions can be transformed with connection to geometric and proportional reasoning. I think that manually manipulating rectangles in the Mondrian designs helps students have a better understanding and more practice while thinking about all the possible sizes, number of squares and portions they get. If my future student finds the concept of fractions confusing, I can guide them do the basic counting of squares to obtain an answer. Now, in reference to the prompt visual representations, my students will have the freedom to explore and make connections with equivalent fractions as well as how and why to simplify fractions.

In comparison with the traditional hands-on activities that generally feature combining parts into a whole and then decomposing that whole back into parts, participants described how the Mondrian-style rectangle sets enabled them to move beyond individual examples of fractions and toward an understanding of the larger abstract concept of fractions. The learning featured activities representing and comparing many fractions simultaneously, and this was one of the distinctive elements that facilitated students to explore fractional relationships. Moreover, the creative process of making Mondrian-style original artworks created an engaging learning environment that may improve students interest in mathematics. One of the participants, Paula described her insights about the mathematics teaching method examined:

> Many hands-on activities like cutting and pasting as well as manipulatives for representing fractions do not give the students a whole representation. Mondrian designs were systematic, it was easy to follow and complete. I would like to say that this is the first time I do something like this, with my own hands. It enabled me to acknowledge the advanced ways that multiple fractions can be represented at the same time and how each fraction is related to the other fractions. It would be much easier for students to first activate their artistic motivation by setting up an enjoyable learning environment, and then to develop their math and logical thinking by connecting art and math. This is where I can see how to guide my students to build their foundational knowledge

> of fraction, a concept which I myself had a negative disposition before but now became more interesting and engaging.

As did Paula, a number of participants reported that their experiences completing the Mondrian-style task improved their awareness of limitations to traditional fraction-teaching approaches, especially from their own past learning experiences. For example, instead of replicating how she was taught as an elementary student, Yasmin said that she is going to change her pedagogy for instructing fractions to her future students. Yasmin noted:

> Many teachers use one and only one representation (i.e., pizza) to teach fraction and [it] can cause future comprehensive conflicts when other teaching approaches are used. I remember fraction being taught straight to the point with just one process to follow to get to the answer. My teachers used to teach us how to conduct fraction operations so we used to memorize the routine process but we never understood the why or the reason behind that. I never had the opportunity to develop critical thinking by exploring through hands-on activities like the Mondrian Arts activity. The Mondrian Arts activity helped me to realize that the same fraction can have different representations, yet their space occupied will always be the same.

Discussion

This research study empirically examined innovative pedagogy that emphasized participants creating and using original visual artworks to support the learning of mathematics. The investigated mathematical tasks were selected to allow the researchers an opportunity to better understand the instructional potential for visually-based fractional and proportional representations during mathematics teaching. Our findings were consistent with previous studies (Buforn & Fernández 2014; Hilton & Hilton 2018; Livy & Vale 2011) that showed preservice teachers were commonly underprepared for instructing their students on the topics of fractions and ratios; likewise, most of the participants in this study self-identified the presence of personal knowledge challenges when analyzing their own understanding of fractions and ratios. Results from this study revealed that when a judiciously combined fractional task and proportional task were provided, many of the participants were able to improve their mathematics content knowledge by transforming previously disconnected concepts into interrelated topics.

Connections between ratios and the fractional values among the rectangles the participants created in their Mondrian-style rectangle sets enabled instructional insights. Specifically, during the first task a part-to-whole relationship was the primary focus during an area comparison contrasting each of the individual rectangles and the pre-defined reference rectangle encompassing six grid squares. However, in the second task, a part-to-part relationship was the major focus during a comparison of the lengths and widths within each individual rectangle. Results from this study showed that most of the participants employed visual-spatial strategies, such as comparing a predetermined "one whole" rectangle that was provided to help them formulate and check their fractional reasoning as they created Mondrian-style rectangles. Consistent with several previous studies (Buforn & Fernández, 2014; Hilton & Hilton, 2018; Livy & Vale, 2011), the results also demonstrated that a substantial percentage of the participating preservice teachers were aware of the limitations to their own mathematics knowledge in regard to understanding fractions and ratios. Post-activity reflections from many of the preservice teachers indicated that the visual supports during the instruction helped them to identify their own mathematical misunderstandings. Thus, in line with Nutov (2021), our findings also indicate that mathematics instruction, especially when introducing concepts with dissimilar symbolic representation but equivalent mathematical values (i.e. improper fractions and mixed fractions), can be effectively supplemented by the appropriate use of visual supports as the bridge to consolidate the knowledge connections.

During data analysis, the researchers noted that the participants' pedagogical knowledge for teaching fractions was often overly reliant on the use of mathematical procedures. This finding was not surprising, since procedural fluency is often a main focus of K-12 math instruction, but this almost exclusive focus on procedure appears to have resulted in the impairment of other, deeper and more impactful approaches to engaging with math topics (Ostler, 2011). As our study's findings illustrated, some of the participants learned how to make connections between their visual perceptions and the abstract mathematical concepts they will be responsible for teaching—and these results provide evidence that mathematics instruction might be more effective if visual supports are systematically employed where appropriate. Overall, the findings from this study suggest that elementary teacher educators and elementary teachers may best serve their students by mixing visual supports with their mathematics pedagogy.

Both national and state-level mathematics teaching standards have historically recommended exposing students to transdisciplinary learning opportunities such as finding connections between mathematics and the arts, as was done in this research project wherein mathematical concepts were embodied through a visually-based format (National Council of Teachers of Mathematics, 2000, 2006). This study highlighted several central issues impacting the effectiveness of teacher education related to the teaching and learning of fractions and ratios. The empirical findings from this study demonstrated that preservice teachers could benefit from additional preparation at using contextualized pedagogical approaches to generate high-quality instruction for teaching fractions and ratios effectively. This approach would also help to remedy any gaps in preservice teachers' content knowledge about fractions and how they are related with ratios and proportions.

Mondrian-style rectangle sets have distinctively embedded geometrical and proportional components, which can offer classroom teachers a convenient and comprehensive teaching resource for enabling students to explore complex proportional relationships within fractions and ratios by using visual supports. Generating Mondrian-style rectangle sets by directing the creative placement of horizontal and vertical lines can result in a mathematical learning resource that effectively utilizes the art creation and appreciation processes. The exceptionally structured and geometrical nature of Mondrian-style art can enable a lush pedagogical context for investigating the differences and similarities between fractions and ratios, assisting both students and their teachers (Weiland et al., 2021). So that the instructional approach described in this study might become inclusive of teaching other mathematical topics in addition to ratios and fractions, the current paper concludes by encouraging further research systematically examining the pedagogical potential and challenges pertaining to the use of visual supports as an effective supplement for teaching K-12 mathematics.

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NOTES FROM THE FIELD

The Spring 2023 issue features two *Notes from the Field* with differing temporal placements. Reiser and Trusnovec first report on an adaptation of the Building Thinking Classrooms (BTC) framework and its benefits when used in a college calculus classroom. Following this, Darrow details the historical trajectory of the standards-based mathematics and assessment movement in the United States.

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NOTES FROM THE FIELD

The Thinking Classroom in a College Setting: A Case Study

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KEYWORDS *mathematics teaching, building thinking classrooms, thinking classrooms, post-secondary, calculus*

Introduction

Peter Liljedahl describes a Building Thinking Classrooms (BTC) framework that shows teachers how to set up a classroom that promotes thinking. BTC is divided into 4 toolkits. The first toolkit consists of thinking tasks, vertical non-permanent surfaces, and visibly random groups. The second pertains to defronting the classroom, giving thinking tasks verbally and early while standing, voluntary homework, and mobilizing knowledge. The third consists of using hints and extensions to maintain flow, consolidating, and students writing meaningful notes. The fourth toolkit involves assessment.

Though Liljedahl's research pertains to K-12 classrooms, the framework sounded promising for higher education as well. We decided to try it in a college-level calculus I course, taught at a small liberal arts college in the spring semester. There were 15 math and computer science students enrolled. The class met twice a week for 2 hours each day over a 13-week semester. There was one professor and two teaching assistants. We applied most of the aspects of the toolkits to see if the framework was upheld in a higher education setting.

Course Design and Implementation

To plan the course, we listed learning goals divided into limits, differentiation, applications of differentiation, and integration (https://tinyurl.com/calculusLearningObjectives). Next, we produced assessment questions for each learning outcome, including an advanced and beginner category.

Next, we designed the lessons. A thinking task is one that promotes students to think rather than mimic the teacher. In a K-12 classroom it is suggested to use the beginning days to do non-curricular tasks so that students can experience a thinking classroom. Since our class met only twice a week, we used the first day for non-curricular tasks and introduced the culture of the classroom. After the first class meeting, we moved on to learning the curriculum. We primarily used thin slicing, giving a list of problems that get progressively more difficult. To launch these, we followed Liljedahl's suggestion of connecting the new topic to previous knowledge. Here is an example dialogue to launch the derivative of log functions, using previous knowledge of derivatives of exponential functions, implicit differentiation, and inverse functions.

Teacher: If $y = e^x$ what is $\frac{dy}{dx}$?

Student: e^x.

Teacher: How are logarithms and exponentials related?

Student: They are inverses.

Teacher: How can we re-write $y = ln\ (x)$ as an exponential? [if needed remind students that ln is the same as $\log_e$].

Student: $e^y = x$.

Teacher: We ultimately want to find $\frac{dy}{dx}$, the derivative of $ln\ (x)$. Since we already know the derivative of an exponential function, what technique can we use here?

Student: Implicit differentiation.

Then, students worked in visibly random groups on vertical whiteboards. The non-permanence of the boards gives students permission to make mistakes. Since the boards are vertical and placed around the classroom, students stand as they work, keeping them more engaged by discouraging anonymity and keeping them physically active. We gave one marker per group to discourage students from working individually.

After students completed the problems, we consolidated their learning by holding a class discussion on the problems and making connections to the learning goal. This was followed by individual time where students practiced and wrote notes.

For assessment, we utilized check (2 points), check minus (1 point), and X (0 points) symbols on their learning goal sheet correlating to a points system. To receive full credit on a learning goal, students must achieve a check in the basic column and 2 checks in the advanced column for that topic. Our main source of assessment was weekly quizzes. Students could retake each quiz at their own discretion. Oftentimes, learning goals were repeated on later quizzes, giving students multiple opportunities to show understanding. The main benefit is that students can achieve every learning goal at any point in the semester.

We also assessed through a portfolio. We set up a portfolio template in Google Slides (https://tinyurl.com/calculusPortfolio). Each week we gave time in class for students to write a reflection in their portfolio. At the end of each unit there was a reflection with questions pertaining to critical thinking, communication, collaboration, and creativity. The final part of the portfolio was for students to show how they achieved each learning goal.

Framework Modifications

One part of the framework we were not able to utilize was defronting the classroom because we had to share our classroom, making it infeasible to change its configuration. One aspect we changed was giving tasks while standing. While students stood most of the class, we realized that with a two-hour class they did need some seated time. We sometimes launched tasks verbally and sometimes used a set of slides, especially if a visual was needed. The framework suggests homework should not be graded or checked. We modified this by using a courseware system to assign homework. The homework did count towards their grade, but students could make as many attempts as they wanted, getting feedback after each attempt.

Student Interest & Engagement

We gauged students' interest in mathematics before, during, and after the semester. On the first day we used non-curricular BTC tasks. Students expressed they were unfamiliar with BTC tasks and described aspects of mimicking behaviors in previous math classes. One student stated, "Most of my math classes were strictly based on memorization rather than learning math itself." We discussed BTC through an interactive syllabus that outlined the framework and expectations alongside reflective questions. We shared the framework with students to justify why we were running the course this way. Subsequently, students felt that sharing their ideas would be the most difficult part because they get nervous or are not good at communication.

A few weeks into the semester, seven out of 15 students volunteered for an 8-10 minute semi-structured interview to discuss the course so far. Four students seemed to prefer a student-centered classroom, two liked a mix, and one preferred lecture because "that is what I am used to." They all had positive things to say about the class environment and learning techniques. Many preferred groupwork over lectures.

Students appreciated that they could learn from their mistakes. One student said, "In my other math classes, there's a really big difference. With this one, if you make a mistake and realize after, you can go back and fix it."

While students felt nervous about sharing their ideas at the start of the semester, many expressed feeling more comfortable doing so several weeks into the course. One student said, "Everyone respects everyone's opinions here and helps each other out."

In the end of semester portfolio, all students made positive comments about working in groups. "Having the ability to work in groups really strengthened my experience with this course. I enjoyed how I was also able to connect with the other students in the class in a positive way."

We monitored and measured student engagement during 13 consecutive classes in the second half of the semester by observing how often students passed the marker, provided input in their group, and sought assistance through intergroup collaboration. Passing the marker showed more than one student was engaged in the problem. Input in their group showed that students were engaging in discussions rather than independently solving problems. Seeking assistance through intergroup collaboration led to students being more reliant on their peers, thus being more engaged. Based on the high participation in these three categories, it was evident that students were engaged through these actions.

Homework

Each homework assignment had a due date, but students were allowed to continue working on them throughout the semester. Thus, providing homework grades as proof of achievement would not be useful, but it is worth noting the homework completion percentages. In this course, homework completion rates were high. Only one student completed less than 93% of the assignments because they stopped all class activities in the last month of the semester.

Achievement

Figure 1 depicts the grade distribution for learning goal achievement in each unit.

The numeric-to-letter grade distribution is as follows: A = 90-100, B = 80-89, C = 70-79, D = 60-69, F = 0-59. That is, students who scored an A in each unit achieved a minimum of 90% of the learning goals, and so on. Many students earned an A for every unit, making us believe that learning truly was present throughout the course.

The last unit on differentiation had the fewest A's, as students did not have as many opportunities to demonstrate their learning on this topic. That also provides evidence that students were taking advantage of the opportunity to get more checks in earlier topics when they had more time. This differs from a traditional classroom, where most students are given one opportunity (typically a unit exam), to demonstrate learning prior to moving on to the next unit.

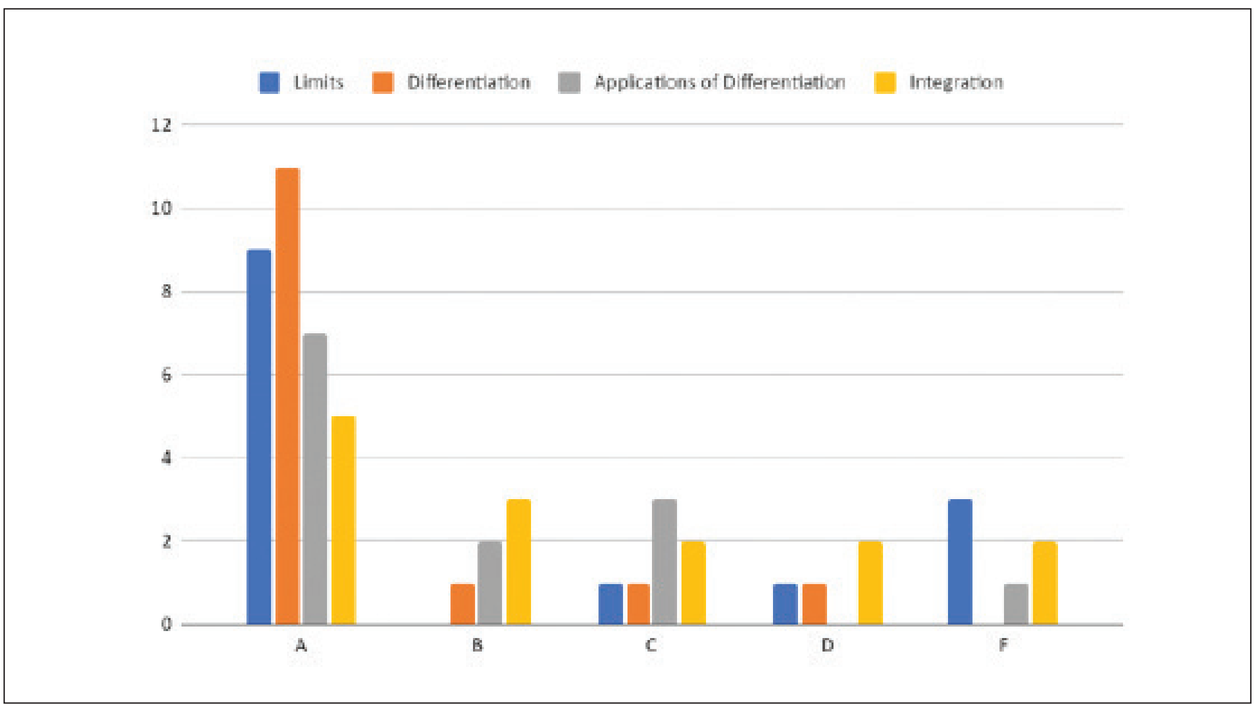
Lessons Learned

There are some areas of BTC that we would like to work on, the first being homework. It would be helpful to clearly state what learning objectives are associated with each homework question and have students keep a log of which learning goals they were able to achieve. Therefore, students take ownership over their learning and become more familiar with the learning goals, making it easier for them to understand our assessment practices.

For the portfolio, we thought the reflections were meaningful. However, it felt as though students were simply complying with this requirement and not thinking of it as a learning tool.

Figure 1

Grade Distributions



We would like to try BTC in additional semesters to get a better understanding of how different students pursuing various majors respond to BTC. We would also like to try this again after the COVID-19 pandemic has subsided.

What Went Well

Instructing a college course with BTC strategies brought forth many positives, including a strong sense of class community, high levels of participation and thinking, and assessment practices that students liked. The classroom community was evidenced by friendships that formed as well as observing interactions between students and comments from the portfolio such as, “everyone respects everyone’s opinions here and helps each other out.” Additionally, in contrast with other courses offered at our college, students consistently participated in class. This was made apparent through observations and discussions with students as we circulated between the groups. Finally, weekly quizzes were well received in this course, with students citing the opportunity to retake quizzes as particularly helpful.

Conclusion

Utilizing BTC in a college setting made student interest in the course content high, led to a high rate of homework completion, brought forth student engagement, made student learning evident, and developed a strong sense of classroom community. Although some students still preferred direct instruction, many students expressed their enjoyment of a student-centered classroom. Therefore, based on our experiences, we believe that the BTC framework can be successfully implemented in a college mathematics setting with appropriate modifications.

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NOTES FROM THE FIELD

Briefly Recalling Some Antecedents of Standards-Based Reform and Standardized Assessment in American Mathematics Education

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KEYWORDS *history of mathematics education, educational reform, standardized assessment, educational standards*

Early Roots of Standardization in Mathematics Education

By the turn of the twentieth century, mathematics education in the United States had been the subject of educational concern for more than a century. Substantial developments in pedagogy and curriculum were sparked by a reevaluation of the teaching and learning of the Colonial Period, which was dominated by the "rules" or "rule method" of teaching which valued core tenets of mental discipline theory (Cohen, 2016; Kliebard, 2004). In the early 1800's, groundbreaking advancements in pedagogy driven by innovative textbook authors challenged the pedagogy of procedural drill and memorization by advocating for a focus on developing conceptual understanding through hands-on, exploratory learning (Bidwell & Clason, 1970; Bjarnadóttir, 2014; Cohen, 2016). This continued well into the Progressive Era of education in the early 1900's, which centered on the holistic development of the individual child to become a fulfilled, productive member of society (Dewey, 1915; Rodgers, 2002). For mathematics education, this translated into a desire to implement a pedagogy that placed value on conceptual understanding and meaningful application by "letting children learn by doing" (Kilpatrick, 2014, p. 329).

Despite significant enthusiasm for such progressive ideals, they were ultimately "rapidly overshadowed by the increasing demand for technological and practical mathematical skills" after the First World War (Permuth & Dalzell, 201, p. 238). As a result, Schoenfeld (2016) notes that by the early 1920's, the focus of mathematics education had once again shifted away from a focus on deep conceptual learning of abstract topics to "the concrete—the arithmetic of home and store" and to "practically oriented applications" (p. 499). According to Schoenfeld, this began a period of "'uniformizing' of curriculum and assessments" which "fed naturally into the measurement regime that typified the first half of the century" (p. 499).

This is recognized as one of the earliest moments in mathematics education where the value of standardization can be clearly seen (Kilpatrick, 2014; Madaus et al., 2003; Schoenfeld, 2016). Schoenfeld (2016) notes that this marked the beginning of "the emergence of scientism" where "'objective' measurement and 'rigorous' methods" began to capture widespread interest (p. 500). Such scientific terms are some of the first historical precursors to modern synonymic terms of assessment and accountability, which are central to the modern standards movement. In modern education, data collection through standardized assessment to evaluate educational outcomes is commonplace. However, during this time period in the 1900's, the stable marriage of scientism and mathematics education was just being formed.

It seems that just as mathematics education was particularly vulnerable to the application of mental discipline theory in the 1800's, so too was it vulnerable to the methods associated with standardized assessment of procedural mathematical knowledge (Cohen, 2016; Kliebard, 2004; Madaus et al., 2003; Schoenfeld, 2014, 2016). Assessments of procedural knowledge that consider only right or wrong answers—not the mathematical processes or cognitive effort required to complete them—are easy to develop, replicate, and standardize. Moreover, many such assessments often do not assess higher cognitive processes associated with true mathematics learning and doing (NCTM, 2000; NRC, 2001; Madaus et al., 2003; Schoenfeld, 1985, 2013; Stein et al., 2000). Thus, the ease with which procedural assessments of mathematics knowledge can be created, administered, and interpreted, coupled with their perceived association with educational outcomes, has created a lasting place for them in the field. As Kilpatrick (1992) notes, such a movement of assessing student performance through standardized tests in mathematics "had begun around 1910 and was in full bloom by the 1920's and 1930s" (p. 138). The significance of this bloom has been widely recognized and is poignantly characterized by Schoenfeld (2016) as a movement that would "plague education research and practice through the entire 20th century and beyond" (Schoenfeld, 2016, p. 500).

World War II, New Math, and Back to Basics

To fully appreciate the formation of the Standards Era, one must first consider the historical backdrop to its formation. As Schoenfeld (2016) has famously written, "Wars—whether hot, cold, or economic—focus attention on the mathematical and scientific preparedness of American's citizenry" (p. 503). World War I, for example, drastically shifted progressive ideals within mathematics education to those valuing uniformity and practicality (Schoenfeld, 2014, 2016). In citing Garrett and Davis (2003), Kilpatrick (2014) notes that World War II "proved to be a pivotal event that revived interest in school mathematics as an area of curricular concern following decades of decline" (p. 330). Many scholars agree that the previous calls for reform in school mathematics were in fact "legitimized by the war" (NCTM, 1947, as cited in Permuth and Dalzell, 2013, p. 238).

The "New Math" era, as it became to be known, was seen as the answer to these calls. The curriculum of New Math centered on the introduction of new applied and abstract mathematical topics; an attempt to establish a greater cohesion and uniformity within the progression of school mathematics topics; a renewed emphasis on the logical foundations of mathematics and the precision of mathematical argument; a focus on supporting instruction that promoted discovery on the part of the students; and a focus on providing students with a greater foundation for the growing scientific nature of the nation's workforce (Fey & Graeber, 2003; Garrett & Davis, 2003; Kilpatrick, 1992, 2014; Schoenfeld, 2014, 2016). Shortly after this time, the success of the Soviet Union in launching Sputnik I in 1957 left Americans with a feeling that the country had lost the international "Space Race", and federal funding through the National Defense Education Act (NDEA) of 1958 was passed in response to improve higher education and the development of students in the scientific disciplines.

Despite the generation of "a great deal of enthusiastic activity throughout the school mathematics community" of the time (Fey & Graeber, 2003, p. 531), the New Math movement was later criticized nationally and is rarely seen as a successful school reform initiative by both historical and modern critics (Fey & Graeber, 2003; Kilpatrick, 2014; Schoenfeld, 2016). Modest successes in the improvement of curriculum and instruction were not enough to quell public criticism (Fey & Graeber, 2003; Kilpatrick, 1992, 2014). As a result, another short-lived, and largely unsuccessful, reactionary movement in mathematics education known as "Back to Basics" was sparked, which partly focused on skills that were aimed at improving college admissions scores (Fey & Graeber, 2003; Kilpatrick, 2014). As Fey & Graeber (2003) note, this movement was stimulated by the public consensus that the movements following the launch of Sputnik, including New Math, were a failure. The Back to Basics movement also saw a reemergence of more traditional pedagogical practices, replacing the forward-thinking discovery-based approach popular of the New Math era, as well as a renewed emphasis on assessment and accountability to evaluate schools and teachers, leading to a "process-product paradigm" where national standardized tests served as evaluative measures, a theme that would continue well into the Standards Era (Fey & Graeber, 2003, p. 541).

A Nation at Risk and the Birth of the Standards Era

After decades of perceived decline in mathematics schooling; failure and abandonment of several reform initiatives; failure to win the "Space Race"; low scores on international assessments of school subjects; and the evolving social conditions influencing schooling in the United States, stakeholders in education had reached a boiling point of dissatisfaction and angst nearing the 1980's (Permuth & Dalzell, 2013). Such a feeling was epitomized and catalyzed by the famous—and infamous—1983 report of the United States National Commission on Excellence in Education (NCEE), *A Nation at Risk: The Imperative for Educational Reform* (NCEE, 1983). Historians and mathematics educators characterize the document as one of the most influential documents in mathematics curricular change for the nearly forty years following its publication (Beck et al., 2002; Ferrini-Mundy, 2000; Kilpatrick, 2014; Schoenfeld, 2014, 2016). Permuth and Dalzell (2013) write that "the extremely influential document created barely controlled panic" (p. 242) through its strong and condemning language and blistering critique of public education.

Almost immediately, calls for improvement and accountability in schools were again sparked across the nation. Discussions of widespread standardization and assessment that had been happening in the background for the past forty years were reappearing at the forefront during this time (Kilpatrick, 2014). The most influential group in mathematics education in the country, the National Council of Teachers of Mathematics (NCTM), ultimately satisfied national demand for a standards-based initiative that would fix the seemingly ailing mathematics education system and alleviate public concern. The group's 1980 *Agenda for Action*, which broadened the aims of mathematics education both from a curricular and professional perspective, took on new meaning and application in the wake of *A Nation at Risk* (Fey & Graeber, 2003).

This was later followed by the 1989 NCTM *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), which served as the first nationally recognized and widely implemented official standards-based document of the Standards Era. Kilpatrick (2014) notes that this document was unique and historically significant since it was produced by a professional organization, rather than a governmental agency without outside funding. Further, it was one of the first documents of its kind that "attempted to go beyond local, state and provincial boundaries in laying out recommendations for curriculum and evaluation" (p. 331) in an especially sensitive time. This seminal document was followed by the *Professional Standards for Teaching Mathematics* (NCTM, 1991) and *Assessment Standards for School Mathematics* (NCTM, 1995), which, when taken together, formed a core structure for standards at all levels of mathematics curriculum, instruction, and assessment. These were later refined and formed the widely implemented next installment of the NCTM Standards, the *NCTM Principles and Standards for School Mathematics* (NCTM, 2000).

The standards set forth by NCTM were historically significant for several reasons. First, they were presumably the first of their kind in mathematics education. Since this time, standards-based initiatives have permeated and become a central component to mathematics education policy, research, and practice (Schoenfeld, 2016). Second, the standards deviated, in a significant way, from previous "top-down" reform initiatives of the 1900's and instead focused on "those very close to decisions about mathematics curriculum—teachers, supervisors, and developers of instructional material" (Ferrini-Mundy, 2000, p. 38). Third, as mentioned previously, the standards transcended local boundaries and became the first nationally recognized curriculum reform document. As a result, this work "took on a life of its own" (Ferrini-Mundy, 2000, p. 38) and began to influence national science standards; the local and state development of additional curricular standards; and were reflected in independent instructional materials and textbooks. This was partly due to the attractive nature of standards in providing a common language for professionals to communicate desired outcomes and adjust their practice.

The Standards and Standardization: Connections to Today

The push for greater accountability coupled with the perceived success of standards-based reform initiatives has fueled a reemergence of scientism and standardized assessment in the United States at all levels of education (Schoenfeld, 2014). Arguably the most influential—and detrimental, according to many scholars—reform initiative in modern education was the 2001 No Child Left Behind Act (NCLB) (Beck et al., 2002; Ferrini-Mundy, 2000; Schoenfeld, 2014, 2016). According to Schoenfeld (2014), NCLB "epitomized" (p. 53) the standards movement and completed the connection between standards-based reform and accountability. The focus on assessment for accountability, Schoenfeld (2014) writes, created the pervasive idea that "students, schools, districts, and states must meet certain standards or suffer the consequences" (p. 53). In addition to the goal of generally improving education, NCLB also targeted the achievement gap, which scholars have notably criticized as an educational focus due its potential to perpetuate negative and inequitable narratives with respect to race and achievement (Gutiérrez, 2008).

Originally aimed at improving American education, NCLB instead became a gatekeeper of federal funding for education and proliferated standardized assessment for accountability. Therefore, the modern era of high-stakes, standardized assessment on which many evaluations of teacher, school, and district performance and the distribution of national funds relies was in many ways cemented with NCLB (Reys, 2014). Moreover, the central yet lofty goal of having 100% of all students be proficient in mathematics by 2014 was not obtained and created substantial anxiety for schools, dishonesty in reporting, and an antithetical lack of consistency and accountability at all levels of education (Resnick et al., 1992; Reys, 2014). NCLB was ultimately deemed a failure and was later replaced by the Every Student Succeeds Act of 2015. However, the lasting effect of NCLB on standards-based assessment for accountability remains.

More recent standards-based initiatives such as the Common Core State Standards for Mathematical Content (CCSSM) have been implemented to improve the quality of mathematics curricula and teaching in an attempt to provide a unified, national set of standards (NGA, 2010). Sponsored by the National Governors Association (NGA) and Council of Chief State School Officers, the CCSSM was aided by a substantial federal investment in its implementation, resulting in the majority of all states adopting the original, or a modified form of the standards (Hill et al., 2019; Porter et al., 2011). The NGA note that the standards were developed to make the mathematics curriculum in the United States "substantially more focused and coherent in order to improve mathematics achievement" as well as "answer the challenge" of "a curriculum that is 'a mile wide and an inch deep'" (NGA, 2010, p. 3).

For some, the CCSSM were notable for building on the successes of the NCTM standards and establishing a state and federally supported push for a national curriculum. However, critics have noted that the CCSSM seems to share the same characteristics of NCLB (Hess & McShane, 2013) and that the CCSSM merely "provides the basis for a new generation of standardized tests" (Tampio, 2018, p. 8). In addition to these tests, in accordance with the Every Students Succeeds Act of 2015, several states require perhaps the most traditionally debated standardized assessment in the United States, the SAT, as a federally required assessment, which has in turn has become an assessment utilized to assess learning with CCSSM. Currently, the growing sentiment is that the initiative has resulted in yet another standards-based installment that, like its predecessors, is difficult to implement, monitor, and consistently modify. More recently, it has been noted that the development of state and local school mathematics standards is effectively "signaling the end of Common Core" (Lee, 2021).

Regardless of the latest reform trend, it currently seems as if standards-based curriculum and standardized assessment are as common in modern schooling as brick and mortar. This is a consequence of their mutual historical development as intimately linked components of educational reform. Since this piece provides only a brief and general discussion of some of the intricate and interrelated factors of this complex history, more comprehensive and informative works on these topics cited here should certainly be consulted for further reading. This paper merely serves to recall several important moments in the history of mathematics education history to provide another lens through which to view modern mathematics education in the United States.

It should also be noted in closing that the efforts of reformers and other stakeholders in education for more than a century have resulted in a great many improvements to the quality of curriculum and instruction in mathematics and have benefitted both the educational enterprise and the children learning within it. However, it is also clear that the waves of educational reform discussed here have, despite best intentions, created a lasting place for standards-based curriculum and standardized assessment in the field of mathematics education. There is growing evidence that the assessment and accountability movement has had detrimental effects on the growth and success of education in the United States, which has even led to a growing countermovement of standardized test refusal (Braun & Marion, 2022; Pérez, 2018; Resnick et al., 1992; Tampio, 2018). Additionally, despite the high value placed on these tests for academic decision making, from evaluations of school districts to college admissions, recent longitudinal studies have shown that characteristics inherently unmeasurable through standardized assessments of crystallized learning are more influential and meaningful predictors of collegiate academic success, retention, and graduation and success in the mathematics classroom (Ben-Avie & Darrow, 2019).

Nevertheless, it is a dubious proposition at best that the current educational enterprise will shed the trappings of assessment and accountability that have become solidified over the course of the past century. From standardized college admissions tests such as the SAT to Advanced Placement Examinations in high school to yearly standardized grade-level examinations across the country, standards-based, standardized assessment continue to be bound within the educational experience in America. Although more work is certainly needed to evaluate both the successes and failures of the past initiatives discussed here, it is perhaps more important to inform future change through the recognition and consultation of the historical development of the current era of educational reform. For if the aim is to meaningfully reform education for future progress, the past to which it is inextricably linked cannot be overlooked.

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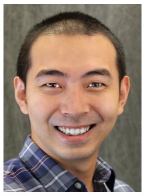

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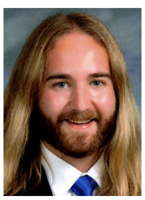

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ACKNOWLEDGEMENT OF REVIEWERS

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This call for papers is an invitation to mathematics education professionals, especially Teachers College students, alumni, and associates, to submit articles describing research, experiments, projects, innovations, or practices in mathematics education. The journal features full reports (approximately 3500 to 4500 words) and short reports (approximately 500 to 1500 words). Full reports describe findings from specific research, experiments, projects, innovations, or practices that contribute to advancing scholarly knowledge in mathematics education. Short reports ("Notes from the Field") provide examples, commentary, and/or dialogue about practices out in the field of mathematics education or mathematics teacher education; examples from classroom experience are encouraged. Although many past issues of *JMETC* focused around a theme, authors are encouraged to submit articles related to any current topic in mathematics education, from which pertinent themes for future issues may be developed. Articles must not have been submitted to or accepted for publication elsewhere. All manuscripts must include an abstract (approximately 150 words in length) and keywords. Manuscripts should be composed in Microsoft Word and follow APA format. Guest editors will send submitted articles to the review panel and facilitate the blind peer-review process. Articles for consideration should be submitted online at jmetc.columbia.edu, and are reviewed on a rolling basis; however, to be considered for the Fall issue, articles should be received by **August 31, 2023**.

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This call for reviewers is an invitation to mathematics educators with experience in reading or writing professional papers to join the review panel for future issues of *JMETC.* Reviewers are expected to complete assigned reviews within three weeks of receipt of the manuscript in order to expedite the publication process. Reviewers are responsible for editorial suggestions, fact and citations review, and identification of similar works that may be helpful to contributors whose submissions appear appropriate for publication. Neither authors' nor reviewers' names and affiliations will be shared with one another; however, reviewers' comments may be sent to contributors of manuscripts to guide revision of manuscripts (without identifying the reviewer). If you wish to be considered for review assignments, please register and indicate your willingness to serve as a reviewer on the journal's website: jmetc.columbia.edu.

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